ELEMENTARY GEOMETRY
PLANE

STONE – MILLIS

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BY

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We ask to-day not what a man knows, but what he can do

PRESIDENT FAUNCE, Brown University

οὐ πόλλ' ἀλλὰ πολύ

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PREFACE

All students of the problems of education are coming to see that geometry as it has been taught traditionally in the secondary schools, while possessing certain educational values, is not well adapted to the present needs and interests of pupils and does not meet the practical conditions of modern life.

A few decades ago geometry was a college subject, and was taught primarily to develop the reasoning powers. In the growth of modern educational institutions, the subject has been forced down into the secondary schools and taught for the same purpose. We are teaching twentieth century boys and girls the subject of geometry with practically the content given it by the Greeks twenty-five hundred years ago, a subject prepared in both form and content for the speculative mind of philosophers and mathematicians. The better adaptation of geometry to the present needs and interests of secondary schools is to be secured in three ways: (1) through *vitalizing the subject* by teaching it in its relation to the practical uses of real life, (2) by *rearranging the subject-matter* according to psychological principles, and (3) through *improved methods of presentation.* It is along these lines of reform that this text-book has been written.

The fundamental features of the book are as follows:

Introduction of Real Applied Problems. — A sufficiently large number of the best of the traditional "original exercises" have been used to meet the demands of the most exacting

school authorities and of college entrance requirements. In addition to these, many of the simpler problems of geometry that are encountered in the various practical fields of human activity have been introduced. This feature of the book presents several advantages: (1) it awakens the *interest* of the pupil in the study of geometry, (2) it supplies a practical *motive* therefor, and (3) it increases the educational value of the subject. In these respects the traditional methods have failed. The pupil using this book will no longer look upon geometry merely as a collection of abstract truths to be mastered for a promised but shadowy mental discipline, but rather as furnishing a *scientific equipment* for the solution of problems actually encountered in practical life. He sees, too, that to make this equipment most effective he must acquire facility in its use by the solution of these same real concrete problems while yet a pupil. As President Faunce says, "Formerly education trained for discipline, later for culture, to-day for efficiency."

Approach through Concrete Exercises, Measurements, and Constructions. — Demonstrative geometry is approached gradually through concrete exercises, measurements, and constructions. Formal demonstrations are not attempted until page 23. A concrete basis is thus furnished for the more formal work that follows. The traditional practice of introducing the pupil to two difficulties at once, viz. dealing with new abstract geometrical concepts, and undertaking to master the niceties of a logical demonstration involving them, is avoided.

The *need* of discovering the truths of geometry through pure reason, and the *nature* of a logical proof, are incidentally developed before the formal demonstration is introduced.

Many of the truths hitherto proved by a line of reasoning too subtle to appeal to the pupil have been assumed at the outset.

The pupil is made proficient in the use of the geometrical instruments, and is early taught how to make many of the simpler constructions of geometry, so that he may use them in accurately drawing the figures involved in the theorems. These constructions are justified later by proofs.

The Book adapted to Second-year Classes. — The features outlined above especially adapt the book to the use of pupils in the second year of preparatory work.

In addition to this, the material is so arranged that the easiest and, as far as possible, the most practical parts of the subject come in the first half of the year's work. This reorganization of the material has been made possible by discarding the useless traditional Greek division of geometry into Books, and organizing the material into chapters according to the modern custom in other books.

Theory of Limits. — The treatment of the *theory of limits* and its application in the proofs of the *incommensurable cases* of theorems is omitted from plane geometry until the mensuration of the circle is reached. Colleges do not require it for admission, and most teachers who attempt to teach the theory of limits to young pupils find it a waste of time.

When incommensurable cases occur in early theorems, the pupil is frankly told that the proof is incomplete and that the consideration of the incommensurable case is purposely omitted as being too difficult for this stage of his progress.

Use of Modern Drawing Instruments. — In addition to the use of the straightedge and compasses to which the Greeks limited the constructions of geometry, the pupil is taught the use of other instruments, such as the triangle and the protractor, as they are used in practical work to-day.

Field Work. — Actual measurements and field work are frequently suggested in the exercises, and will prove both interesting and valuable.

Correlation with Algebra. — In conformity with the modern movement to unify the different subjects of secondary

mathematics, a number of geometrical exercises are given for algebraic solution. The algebraic treatment of theorems has been liberally used.

Trigonometric Ratios. — After studying the relation between the corresponding sides of similar triangles, the pupil is shown how these relations grow into the trigonometric sine, cosine, and tangent. Their applications in computing heights and distances are then learned.

New Geometrical Terms. — Such terms as "sect," "ray," etc., which have been agreed upon by mathematical societies in recent years, have been used in the belief that they will help to clarify geometrical thinking.

Proofs of Many Theorems left to the Pupil. — Instead of giving the complete proofs of all theorems presented as has been generally customary, suggestions only are given in a large number, and the formal proof is left to the pupil. This is designed to stimulate thought and to prevent the too common tendency to memorize the proofs mechanically.

The authors take pleasure in acknowledging their indebtedness to the large number of teachers and educators who have read a part or all of the book in proof and offered many helpful suggestions. They are especially indebted to Dr. Charles S. Chapin, Principal of the Montclair Normal School; William Fuller, Mechanic Arts High School, Boston; Islay F. McCormick, Roxbury Latin School; G. W. Earle, Hyde Park, Mass., High School; Miss Elizabeth G. Hoyt, Providence Classical School; Miss E. Sue MacWilliams, Salem, Mass., High School; Walter A. Robinson, Public Latin School, Boston; A. A. Dodd, Manual Training High School, Kansas City; Prof. G. H. Stempel, University of Indiana; and Miss Alice Lord, Portland, Me., High School.

J. C. S.
J. F. M.

May, 1910.

SUGGESTIONS TO TEACHERS

Material Equipment. — Each pupil should be provided with a straightedge, compasses, protractor, and triangle at the beginning of the work. These may be obtained at a trifling cost. Pupils should be required to make all drawings neatly and accurately.

Constructive Work. — Many problems of construction are given. This phase of the work needs much greater emphasis than has commonly been given it. Many of the constructions of the exercises are of a practical nature. No pupil is expected to do all the constructive work of the text. The teacher will assign only what time will permit. For further suggestions upon what may be omitted, see the next section.

The Use of the Exercises. — No pupil is expected to work all the exercises given. Local conditions and the special needs of the class will determine how much of the material is to be used. *Certain exercises and topics are marked *. These may be safely omitted without interfering with the continuity of the subject if there is not time enough for them.* The material thus marked, however, is not to be considered as of inferior value. This material is of three general types: (1) practical constructive work requiring considerable time, (2) measurements and field work that may not be suited for certain schools, and (3) difficult work that may be reserved for a review of the subject.

Pupils differ in their capacity for work, particularly in mathematics. It is the ideal to provide means for getting from each pupil the best of which he is capable, both in quantity and quality. Those who are able to work at a faster

rate than the rest of the class should be assigned supplementary work from the more difficult exercises of the book.

The Proofs. — As stated in a note in the text, in many theorems the steps of the proof are given, but the reasons are purposely omitted. The pupil is expected to supply them in every case. They are omitted to prevent memorizing and to arouse self-activity. To provide a further training in self-reliance, the proofs of a large number of the theorems are left, with suggestions, to the pupil. He should carefully write out such proofs in steps, giving the reason for every step.

CONTENTS

SYMBOLS AND ABBREVIATIONS

(For reference)

i.e., that is.
$\because$, since.
$\therefore$, therefore.
$\angle$, $\measuredangle$s, angle, angles.
$\triangle$, $\triangle$s, triangle, triangles.
$\square$, $\square$s, parallelogram, parallelograms.
rt. $\angle$, rt. $\angle$s, right angle, right angles.
st. $\angle$, st. $\angle$s, straight angle, straight angles.
rt. $\triangle$, rt. $\triangle$s, right triangle, right triangles.
$\odot$, $\odot$s, circle, circles.
$=$, is equal, or equivalent, to.
$\equiv$, is identical with.
$\sim$, is similar to.
$\cong$, is congruent to.
$>$, is greater than.
$<$, is less than.
$\neq$, is not equal to, *i.e.* $>$ or $<$.
$\ngtr$, is not greater than, *i.e.* $=$ or $<$.
$\nless$, is not less than, *i.e.* $=$ or $>$.
$\perp$, is perpendicular to, or a perpendicular.
$\parallel$, is parallel to, or a parallel.
....., and so on.
$\doteq$, approaches as a limit.

NOTE. — The above take the plural also; thus, $=$ means *are* equal, as well as *is* equal.

ax., axiom.
th., theorem.
def., definition.
cor., corollary.
alt., alternate.
ext., exterior.
int., interior.
hyp., by hypothesis.

PLANE GEOMETRY

CHAPTER I

FUNDAMENTAL NOTIONS

1. **The meaning and uses of geometry.** — To measure a piece of land, a surveyor uses instruments and measures distances and directions. Then, by computations and diagrams, he is able to determine the exact size and shape of the land. The measurements, the computations, and the construction of the diagrams require a knowledge of the science called **geometry.** The earliest people known to have attempted to measure their land accurately were the ancient Egyptians. It is said that because from time to time the overflow of the Nile washed away their landmarks and some of their land, it became necessary to have the land frequently measured in order to determine the just distribution of taxes. The Egyptians, also, must have had other knowledge of measurements in order to construct their pyramids. Many Greek scholars traveled and studied in this land of the pyramids, learned the Egyptian art of measurement, and made it into a science which they called *geometry*, meaning *land* or *earth measurement*.

2. **A wider use of geometry.** — Geometry has come to have a much wider use than merely in land measurement, or surveying. It is used by the civil engineer in the construction of streets, and in the construction of railroads and bridges. It is used in the construction of all kinds of machinery and

buildings. It is used by the artist in his work; by the astronomer in the study of the motions, sizes, and positions of the planets; by the sailor in navigation; by the stonemason in the cutting and economical use of his material. In short, a knowledge of geometry is needed in a vast number of the modern activities of man.

3. Solids, surfaces, lines, and points. — Any object, such as a book, a ball, a block of wood, or the earth, occupies space. In geometry we are not concerned with the substance of the object, but rather with the properties of the room or **space** occupied by it.

The space occupied by any object is called a **geometrical solid.**

Every portion of space, or a geometrical solid, is separated from the adjacent or neighboring space by a **surface.**

A portion of a surface is separated from the remainder of the surface by a **line.**

A line is separated into two portions by a **point.**

4. The generation of lines, surfaces, and solids. — The chief characteristic of a point is that it has *position.* If a point were caused to move, it would *describe a line* as its path. Similarly, if a line were to move in any way, except along itself, it would *describe a surface.* And if a surface were to move in any way, except along itself, it would *describe a solid.*

5. Plane surfaces. — A *flat surface*, such as the floor, the blackboard, or the top of your desk, is called a **plane surface,** or merely a **plane.**[1] We shall confine our work, for the present, to finding out what we can about those lines and points which lie in a *plane.*

[1] See Solid Geometry for a more complete definition of a plane.

THE STRAIGHT LINE

6. The representation of a point or line. — Since a point has no substance, it is invisible. It may be represented to the eye by a small dot or cross. It may be named by putting a capital letter near the dot or cross. Thus, we may speak of "the point A" or "point B." Since a line has no substance, it is invisible, but is represented to the eye by some mark. It is not the lines nor points, then, that we see, but *the physical representations of them.*

· A
× B

7. To draw a straight line. — A **straight line** is made by placing a *ruler* (also called a *straightedge*) upon a plane surface and drawing a pencil along the edge of the ruler.

A line which is not straight, but which is made up of parts all of which are straight, is a **broken line**. A line no part of which is straight is a **curve**.

NOTE. — Since a line has no breadth, the mark which represents it should be made as fine and uniform as possible. Use a hard pencil sharpened to a chisel edge rather than to a point, when ruling a line. Heavy lines are used in the text to relieve the strain upon the eyes.

By folding a sheet of paper and creasing the edge, a straightedge may be obtained. Fold paper to form a straightedge.

Draw several straight lines through a point P. Locate two points, A and B, and connect them by a straight line. Can you draw any other straight line between these two points? Can you extend the line beyond B? How far? Can you extend it beyond A? How far?

Draw two straight lines intersecting in a point. Can two straight lines ever intersect in more than one point? What is the shortest distance between two points?

8. Observations. — From the observations in § 7 we shall hereafter assume that:

I. *Any number of straight lines can be drawn through a point.*

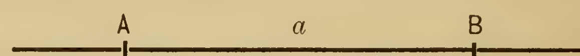
II. *One and only one straight line can be drawn through two points. Or, two points determine a straight line.*

III. *The length of that portion of a straight line between two points is the shortest distance between the two points.*

IV. *A straight line can be produced indefinitely through either extremity. Or, a straight line is unlimited in extent.*

V. *Two straight lines can never intersect in more than one point. Or, two intersecting straight lines determine a point.*

9. Naming a line and a portion of a line. — A line may be named by writing a capital letter at each of two points on it, or by a single letter (usually a small letter is used) somewhere along it.



Thus, we may say "line AB," or "line a."

When a line is read by naming two points on it, an unlimited line is understood. If we wish to restrict our thoughts to the definite part limited by the two points, rather than to the whole line, we shall speak of the **sect**. Thus, by "line AB" is meant the line *through* points A and B, *with no reference to extent;* by "sect AB" is meant *the definite portion* of line AB, *terminating* at the points A and B.

That portion of the line extending indefinitely *from* a point is a **ray**. Thus, the "ray AB" is that portion of line AB extending indefinitely from A through B, while "ray BA" is the part extending indefinitely from B through A.

QUESTIONS AND EXERCISES

1. If the edge of your ruler (straightedge) passes through two points on any straight line, will the edge coincide throughout with the line? Upon what assumption of § 8 is your answer based?

2. Trace the sect AB on a piece of tracing paper, marking points A and B. Turn it face downward so that A falls upon A and B upon B. Is the sect straight? Why?

3. Test the accuracy of your ruler by marking along the edge, then reversing the ruler, and with the edge coinciding with two points of the

line thus drawn, retracing it. What will show whether the ruler is true or not?

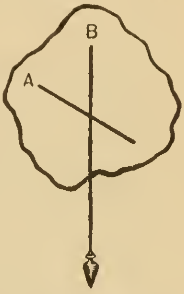
4. Explain what is meant by the expression, "Two points determine a straight line."

5. Three points not in the same straight line will determine how many straight lines? Draw them.

6. Four points in the same plane, no three in the same straight line, will determine how many straight lines? Draw them.

7. What does the expression, "Two straight lines determine a point," mean?

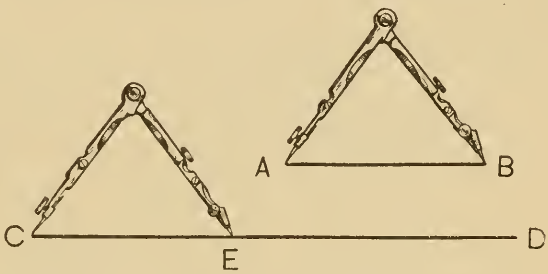
8. How could you locate a point known to be upon each of two lines?



9. If any irregular flat body is supported on a pivot, it will revolve until the center of its mass falls vertically below the point of support. Show that by thus supporting it, first at a point A, then at a second point B, and using a plumb line to mark the vertical lines, the center of mass can be determined.

10.* Use the method of Exercise 9 to find the center of mass of some irregular body. Try to balance it on a pivot at this point.

10. Use of compasses. — A pair of **compasses** (also called **dividers**) may be used to lay off distances on lines and to compare lengths, as follows: To mark off a length on CD equal to the length AB, first place one leg of the compasses at A, then by adjusting the compasses make the other leg fall at B. Then, without changing the adjustment, place one leg at C and cut off the sect CE from the ray CD.



Note. — Compasses are used for drawing arcs of circles, or complete circles, as will be found later.

EXERCISES

1. How can the lengths of two sects be compared by the use of the compasses? In the figure of § 10, which is longer, sect *AB* or sect *CD*? What shows the difference between sects *AB* and *CD*?

2. Draw a sect. Now on a given line lay off a sect twice as long. One three times as long.

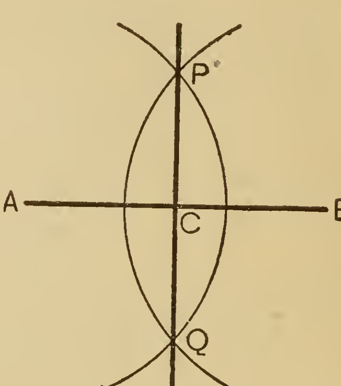
3. On a straight line lay off a sect equal in length to the sum of two given sects.

4. Show that the sum of the sects *AC* and *CB* of the figure in the margin is greater than *AB*.

5. Show that *AC* is greater than the difference between *AB* and *CB*.

Suggestion. — With one leg of the compasses at *B*, make the proper adjustment, and without moving the leg at *B*, show the difference between *AB* and *BC*. Now with the leg of the compasses at *A* and with the proper adjustment, compare this difference and *AC*. Avoid all unnecessary moves and adjustments of the compasses. Each move or adjustment makes possible an error.

11. To bisect a sect. — Any sect *AB* may be **bisected** (cut into two equal parts) as follows: With the compasses conveniently opened, place one leg at *A*, and with the other draw an arc. Then, without changing the opening of the compasses, place one leg at *B*, and with the other describe an arc cutting the first one. Let *P* and *Q* be the points where the two arcs intersect (cross). Now the straight line through *P* and *Q* will bisect the sect *AB*. That is, *AC* and *CB* are equal.



Caution. — Use great care when drawing a line through two points or when putting a leg of the compasses upon a point. Before actually drawing the line, try your pencil over the point and shift your ruler until you are sure the line will pass through the points. Another good way is first to place

your pencil upon the point, then bring the ruler up to the pencil. *Always make very fine, uniform lines.*

EXERCISES

1. With your compasses, compare sects AC and CB in the figure of § 11.

2. In drawing two arcs to determine the two points upon the line bisecting a given sect, the compasses must be opened at least how far?

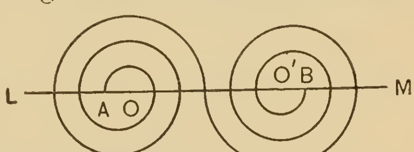
3. Draw sects of different lengths and bisect each. Test the accuracy of the work by use of the compasses.

4. Divide a sect into four equal parts.

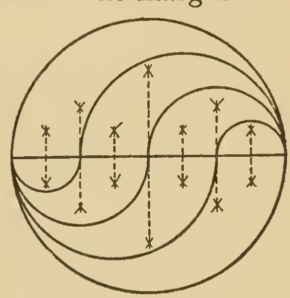
SUGGESTION. — First bisect the sect, then bisect each of its parts.

5. Divide a sect into eight equal parts.

6. Can you use the method of bisecting a sect to divide it into six equal parts? Why not? Into what can a sect be divided by this method?



7.* On a line LM take a sect AB. Divide it into 8 equal parts. With your compasses make an ornamental scroll as shown in the margin.



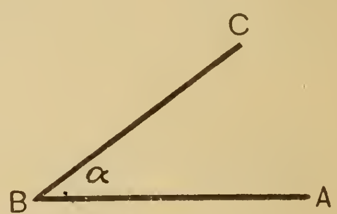
8.* Draw a sect of any convenient length, and upon it construct a design similar to the one in the figure.

SUGGESTION. — Divide the sect into eight equal sects, and use the points of division as centers of the arcs.

CHAPTER II

ANGLES, PERPENDICULARS, AND PARALLELS

12. **An angle.** — Two rays from a common point form an **angle.** The rays are the **sides** or **arms** of the angle, and the common point is the **vertex.** Thus, BA and BC are the *sides* of the angle here represented and B is the *vertex.*



13. **Naming an angle.** — The angle here represented may be named "angle α," "angle B," or "angle ABC." *When using three letters to name the angle, the letter at the vertex is always read between the other two.*

In writing the name of an angle, the symbol $\angle$ is used in place of the word "angle." Thus, angle ABC is written $\angle ABC$.

NOTE. — The Greek letters α (alpha), β (beta), γ (gamma), and δ (delta), etc., are often used in naming angles. They will be used to name angles only, hence will be used without the angle sign. The sign will be used when Roman letters are used. Thus, α, $\angle B$, $\angle ABC$ may denote the angle in the margin.

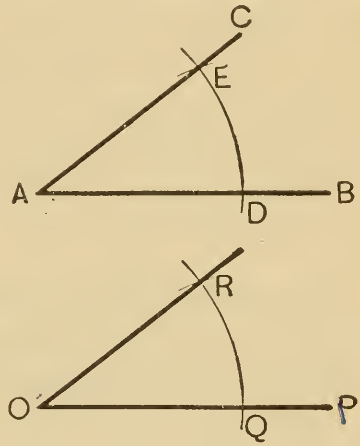
Draw three angles and name each in three ways, using different letters.

14. **The magnitude of an angle.** — The **size** or **magnitude** of an angle depends upon the relative *directions* of its sides, not upon the lengths represented. The sides are rays and are unlimited in extent. Thus, β is larger than α, although equal lengths of the sides are represented.

The sizes of two angles may be compared by tracing one on tracing paper and placing it upon the other so that their vertices are together and a side of one falls upon a side of the other, and observing the positions of the other sides.

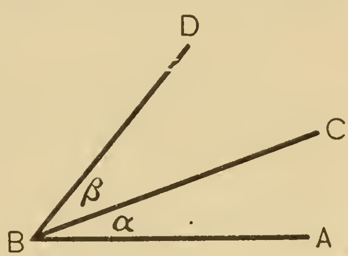
15. To construct an angle equal to a given angle. — An angle may be drawn equal to a given angle as follows:

Given angle BAC. Draw OP. Now with the compasses opened conveniently, and with one leg at vertex A of the given angle, draw an arc cutting the sides of the given angle at the points D and E, respectively. Now with one leg at O, and without changing the opening of the compasses, draw an arc cutting the ray OP at point Q. Now place one leg of the compasses at D and adjust the compasses until the other leg falls at E. Without changing the adjustment, place one leg at Q, and with the other draw an arc cutting the first arc at a point R. Draw OR. Then $\angle POR = \angle BAC$.



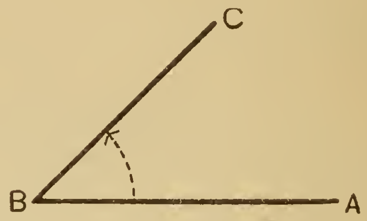
Test the equality of the two angles by the method of § 14.

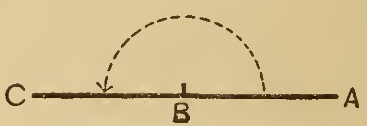
16. Adjacent angles. — When two angles have a common vertex and a common side, but lie upon opposite sides of the common side, they are **adjacent angles**. The whole angle formed by the two adjacent angles is their **sum**. Thus, α and β are *adjacent*.



What angle is their sum? What is the difference between $\angle ABD$ and α? Between $\angle ABD$ and β?

17. A straight angle. — An angle may be conceived as being described by the revolution of an arm counter-clockwise about its vertex from the position of one arm to that of the other. Thus, if the arm BC revolves about B from the position BA to the position BC of the figure, it describes $\angle ABC$. If BC revolves to a position so that the two arms form a straight line, the angle is a **straight angle.**





We shall assume, then, from the definition of a straight angle that:

If two adjacent angles are together equal to a straight angle, their exterior sides form a straight line.

18. A right angle. — If a line be drawn dividing a straight angle into two equal parts, these parts are **right angles,** *i.e.*:

The half of a straight angle is a right angle.

An angle less than a right angle is **acute.** An angle greater than a right angle and less than a straight angle is **obtuse.**

EXERCISES

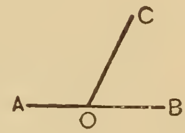
1. Draw two angles. Name one $\angle ABC$ and the other $\angle DEF$. Cut out or trace $\angle DEF$ and place it upon $\angle ABC$ so that E falls upon B and ED falls along BA. If EF falls within $\angle ABC$, which angle is greater? What angle shows the difference? If EF falls along BC, what is known of the two angles?

2. Draw an angle. Name it $\angle ABC$. Draw another angle equal to the first angle. Test the accuracy of the construction by the method of § 14. Repeat this with several different angles until your constructions are accurate.

3. Draw two adjacent angles. Name one $\angle ABC$ and the other $\angle CBD$. What is the common vertex? What is the common side? What is their sum?

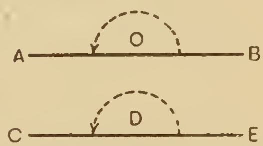
4. Name the two sides of the *straight angle* of § 17. What is the vertex?

5. If from a point O on a straight line AB, a ray OC be drawn, name the two adjacent angles thus formed. What is their sum?



6. If several rays be drawn from O, a point in the line AB, what can you say of the sum of all the angles at the point O on one side of AB?

7. If you should trace the straight angle EDC and place vertex D upon vertex O, of straight angle BOA, and let DE fall along OB, where would DC fall? Why do you know this without actually tracing the one and placing it upon the other? (See § 8, II.)



8. From Exercise 7, you would assume that all straight angles are equal. What, then, can you say of the magnitude of any two right angles? Why?

19. Axioms. — In answering Exercise 8 of the preceding exercises, use was made of the fact that "halves of equals are equal." Those statements about quantities in general which are assumed as true without proof are called **axioms**. The student is familiar with several axioms used in algebra. Some axioms are stated here that will be referred to from time to time in the geometry. They should be given as the authority for such conclusions as the answer to Exercise 8.

I. *Things which are equal to the same thing, or to equal things, are equal to each other.*

II. *If equals are added to equals, the sums are equal.*

III. *If equals are subtracted from equals, the remainders are equal.*

IV. *If equals are multiplied by equals, the products are equal.*

V. *If equals are divided by equals, the quotients are equal.*

VI. *If equals are added to or subtracted from unequals, the results are unequal in the same order.*

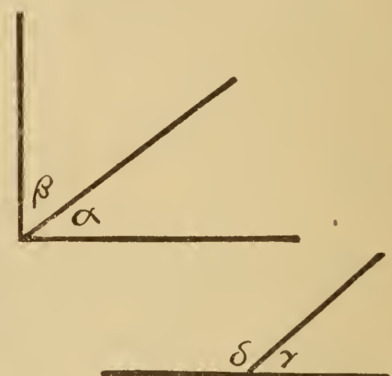
VII. *The whole of a thing is equal to the sum of all its parts, and is greater than any one of its parts alone.*

20. Geometrical assumptions. — Besides axioms, there are purely geometrical truths which we assume. (See § 8.) From Exercise 7, page 11, we assume that:

I. *All straight angles are equal.*

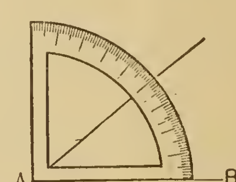
From this it follows by Ax. V that (see Ex. 8, p. 11):

II. *All right angles are equal.*



21. Complementary and supplementary angles. — Two angles whose sum is a *right angle* are **complementary angles**, or one is the **complement** of the other. Two angles whose sum is a *straight angle* are **supplementary angles**, or one is the **supplement** of the other.

Name two complementary angles in the figure. Name two supplementary angles. What is the supplement of δ? What is the complement of α?

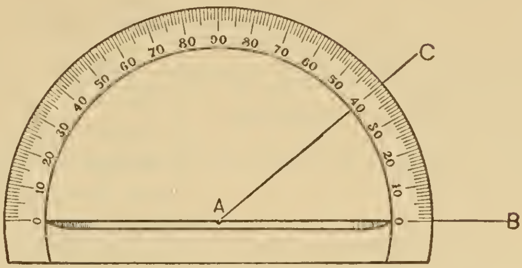


A Quadrant

22. The measure of an angle. — If a right angle is divided into 90 equal parts, each of these parts is called a **degree**. The size of an angle is often expressed in degrees, marked (°). A degree is also divided into 60 minutes (60′) and a minute into 60 seconds (60″).

A **protractor** or a **quadrant** is generally used to measure the number of degrees in an angle. To use either instrument, the center A is placed upon the vertex of the angle,

and AB made to fall upon one side of the angle. The position of the other side shows the size of the angle.



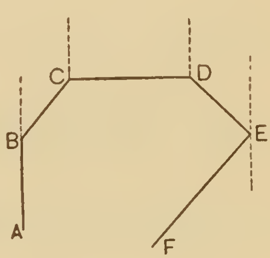
A Protractor

EXERCISES

1.* This table contains the field notes of the survey of the line $ABCDEF$ by some land surveyors, beginning at A. The map of the survey is drawn to scale, 800 ft. being represented by a sect 1 inch long.

Points	Bearings	Distances
B	N.	300′
C	N. 40° E.	250′
D	E.	400′
E	S. 45° E.	255′
F	S. 42° W.	500′

Note. — By *bearings* in the table is meant the directions of the parts of the line as determined by the angles they make with true north and south or east and west. Thus, N. 40° E. means 40° east of north. The sign (′) stands for feet.



Draw a copy of the map, using a scale of 200 ft. to the inch. Use the protractor in turning off the angles at B, C, etc. In turning off the angles first draw a north and south line through each point as it is reached.

2.* These are the notes for part of the survey of a branch of a railroad track, beginning at the point where the new track is to leave the old track. Lay out the map of the new road, using straightedge and protractor, to the scale of 1000 ft. to the inch.

Stations	Bearings	Distances
1	N. 20° E.	500′
2	N. 40° E.	1000′
3	N. 60° E.	1000′
4	N. 45° E.	1500′
5	E.	2000′

3.* These are the notes of a survey of a river bank. Make a drawing to scale, showing the course of the river, and mark the positions of the objects on it.

Points	Bearings	Distances
Rail Fence	N.	800′
Mouth Cat Creek	N. 24° W.	2400′
Old Mill	N. 35° E.	1000′
Poplar Tree	N. 20° E.	600′
Bridge	E.	2000′
Sch. House	S. 80° E.	1600′
Stone Wall	N. 45° E.	2000′

4. How many degrees in a right angle? in a straight angle?

5. If a right angle is divided into two parts, and one part is 60°, how many degrees in the other? If one part is 10°, how many degrees in the other?

6. If a right angle is divided into two equal parts, how many degrees in each part? If divided into three equal parts, how many degrees in each?

7. Two angles are complementary, and one contains 30°. How many degrees in the other?

8. What is the complement of a 70° angle? of a 40° angle? of a 20° angle?

9. If two angles are supplementary and one is 80°, what is the other?

10. If one of two angles composing a right angle is twice the other, how many degrees in each? (Form an equation.)

11. The sum of two angles is a right angle, and one is 6° more than six times the other. How many degrees in each?

12. If $x°$ and 40° are complementary, how many degrees in x?

13. If a certain angle equals one third of its supplement, what part of a right angle is it?

14. What is that angle whose supplement is seven times its complement?

15. State in the form of an equation that α and β are supplementary; that they are complementary.

16. The difference between two complementary angles is 10°. Find the angles.

17. Two angles are complementary. Three times the smaller is 5° less than twice the larger. Find each.

18. One part of a straight angle is 100°. How many degrees in the other part?

19. What is the supplement of an angle of 120°? of an angle of 50°? of an angle of 140°?

20. One of two supplementary angles is five times the other. How many degrees in each?

21. Two supplementary angles have the ratio of 2 to 7. Find them.

22. The angles a and $3a$ are complementary. How many degrees in each?

23. Angles a and $3a$ are supplementary. How many degrees in each?

24. The difference between two supplementary angles is 40°. How many degrees in each?

25. Find the complement of 30° 27′. The supplement of 8° 42′.

26. Find the complement of 27° 13′ 54″. The supplement of 108° 27′ 56″.

27. If both a and β are complements of 60°, how many degrees in each?

28. Angles a and β are both complements of the same angle. Compare angles a and β.

29. a and β are both supplements of 80°. How many degrees in each?

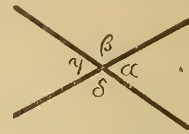
30. a and β are both supplements of the same angle. Compare a and β.

23. Two important inferences. — From the answers to Exercises 28 and 30, of § 22, we may state that:

I. *Complements of the same or of equal angles are equal.*

II. *Supplements of the same or of equal angles are equal.*

24. Vertical angles. — Two lines which intersect form four angles. The two pairs of non-adjacent angles are **opposite** or **vertical angles.** Thus, α and γ are *vertical angles.* Name another pair of vertical angles.



25. Vertical angles compared. — How many degrees in the sum of angles α and β? In $\beta+\gamma$? In $\gamma+\delta$? In $\delta+\alpha$?

Name two angles that are each supplements of β. What, then, is the relation of α to γ? Why? Compare β with δ. Give the reason for your answer.

The answers to the above questions lead to the following important conclusion:

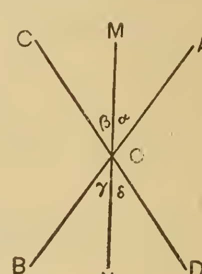
If two lines intersect, the vertical angles thus formed are equal.

Observe that the above truth was derived through a *course of reasoning* rather than by tracing one angle and applying it to the other. Most truths of geometry are derived in this way.

EXERCISES

1. If α (figure, § 24) is 35°, how many degrees in each of the other three angles?

2. If three straight lines meet in a point, how many angles are formed? Of these, how many is it necessary actually to measure with the protractor in order to determine the size of each?

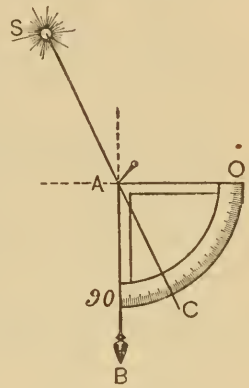


3. The lines AB and CD form two vertical angles, $\angle AOC$ and $\angle BOD$. MN is drawn through O, making $\alpha=\beta$. Compare γ and α, δ and β. Then since $\alpha=\beta$, what must be the relation between γ and δ? See Axiom I, § 19.

4. If lines are drawn through a point on a straight line, making all the angles on one side of the line equal, what relation would exist between the angles formed on the other side of the line? What axiom applies to the angles?

5. If $\alpha : \beta = 2 : 3$ (figure, § 24), how many degrees in each? How many in γ? In δ?

6. If three angles in the ratio of $1 : 2 : 3$ cover all the plane around a point, how many degrees in each angle?

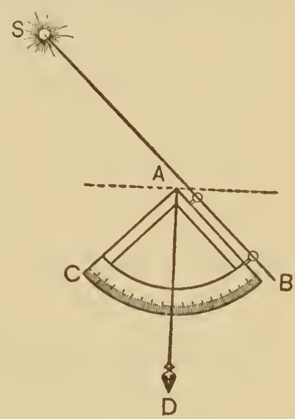


7. If one of the four angles formed by two intersecting lines is equal to 90°, what is each of the other three?

8. The angle of elevation of the sun may be determined as follows: A quadrant is held in a vertical position. A plumb line is fastened to a pin at the vertex of the angle and the quadrant turned until the plumb line AB falls upon 90°. The shadow AC of the pin shows the angle of elevation. Why?

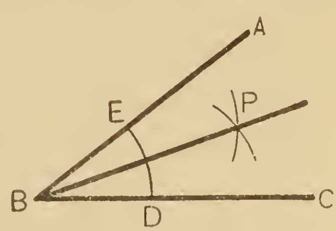
9.* Find the angle of elevation of the sun on the day you study this.

10. *Tycho Brahe* (1546–1601), a Danish noble who built and operated the first astronomical observatory, in his earliest observations used a quadrant for measuring the altitudes of stars, or their angular distances above the horizon. Show that when the instrument was held in a vertical plane and the sights A and B aligned with the star S, the altitude of the star was determined by observing the angle CAD.



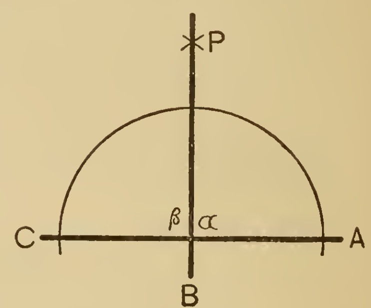
11.* Use the method of Exercise 10 to find the elevations of objects in the neighborhood, as trees, hills, steeples, telephone poles, etc.

26. To bisect an angle. — Any angle may be bisected (divided into two equal parts) as follows: With the compasses conveniently opened, place one leg at the vertex B and with the other draw an arc cutting both sides. Let the points of cutting be D and E. Now with the compasses opened at any convenient angle, place one leg at D and draw an arc with the other. Then, with-



out changing the adjustment of the compasses, place one leg at E and draw an arc cutting the other. Let the point of cutting be P. Draw the ray BP. Then $\angle ABP = \angle PBC$. Test the equality of the two by tracing one and applying it to the other.

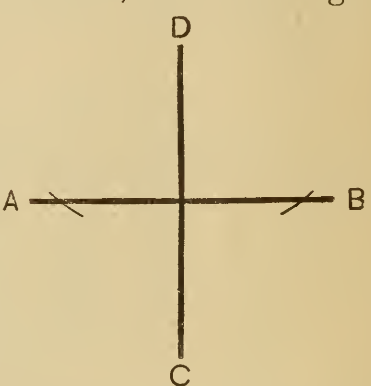
27. Perpendiculars. — We may bisect a straight angle by the method of § 26. Describe the work done in the figure in the margin. What kind of angles are α and β? Why?



When two straight lines meet, forming *right angles*, the lines are **perpendicular** to each other.

In written work the symbol $\perp$ is often used in place of the words "perpendicular to" or "is perpendicular to." Thus, in the figure, $PB \perp CA$.

28. To draw a perpendicular to a line. — It is necessary often to draw a line perpendicular to another, and make it go through a certain point. There are evidently two cases:



I. *When the point is on the line*, we proceed as in § 27. Describe the process and draw perpendiculars from points on a line.

II. *When the point is not on the given line*, we place one leg of the dividers at D (the given point) and draw an arc cutting the given line at two points, as A and B; then proceed as in bisecting a sect. Describe the complete process and draw a perpendicular from a point to a line. How can you check your work?

Assumption. — *It is evident that only one perpendicular can be drawn to a line through a given point.*

The point where a perpendicular to a line meets the line is the **foot** of the perpendicular.

29. Distance. — Draw a line. Take a point P outside of the line and through it draw a perpendicular to the line. Draw any other line through the point. Compare the two sects from P to the line.

The length of the perpendicular from a point to a line is the DISTANCE *from the point to the line.*

Assumption. — *The perpendicular is the shortest sect that can be drawn from a point to a line.*

30. Parallel lines. — Draw a straight line, and mark two points on it. At each of the points draw a line perpendicular to this line. Will the perpendiculars meet at any point? If they did, would the assumption in § 28 be true?

Two straight lines of the same plane that do not meet are called **parallel lines.**

It may be inferred that:

Two straight lines each perpendicular to the same straight line are parallel.

The upper and lower edges of this page are parallel. Point out other parallel lines.

In writing the statement that two lines are parallel, the words "parallel to" or "is parallel to," are expressed by the symbol ∥. Thus, AB is parallel to CD is expressed by $AB \parallel CD$.

EXERCISES

1. Draw an angle and bisect it. How can you check your work? Check it.

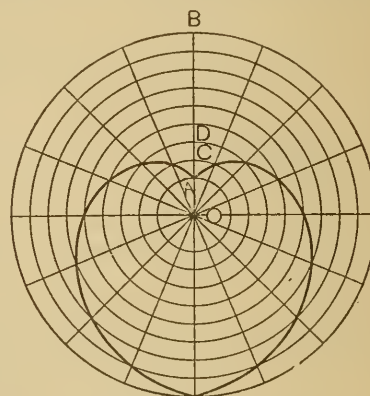
2. Continue bisecting angles until your work is accurate and rapid.

3. Draw a right angle, using only the compasses and straightedge. Draw an angle of 45°. Draw one of $22\frac{1}{2}$°. Draw an angle of $67\frac{1}{2}$°. Test the accuracy of your constructions with a protractor.

4. Divide a given angle into four equal parts. Into eight equal parts.

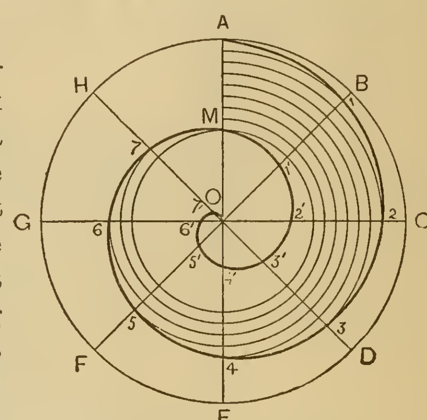
5. Locate a point and divide the angular magnitude around it into sixteen equal angles.

6.* In many different machines, such as the sewing machine, printing press, etc., there is a wheel called a *cam* which is used to modify the motion of the machinery. Cams are constructed in various shapes and dimensions, depending upon the use they are designed for. The figure shows the method of drawing the pattern of a heart-shaped or "uniform-motion" cam. Let the "throw" be AB and the center O. Divide AB into eight equal parts at C, D, etc. Through A, C, D, $\cdots$, B, draw circles with centers at O. Draw rays dividing the angular magnitude around O into sixteen equal angles. Beginning at A, mark the points where the consecutive circles and consecutive rays intersect, and through these points draw a smooth curve, as in the figure.



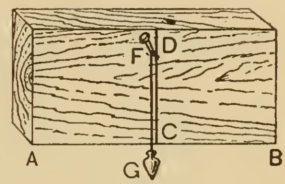
Let the student draw such a cam with AB equal to a given sect m and OA equal to a given sect n.

7.* The Archimedian spiral, discovered by Archimedes, a great Greek mathematician, about 200 B.C., is a curve drawn as follows: If OA is the radius of a circle, the curve starts at A, makes a number of turns about the center, and ends at the center O. If it is to have two turns, begin by drawing radii dividing the angle about O into any number of equal parts, say eight. Bisect AO at M. Divide AM into eight equal parts, and through the points of division draw circles with centers at O, the consecutive circles meeting the consecutive radii at the points 1, 2, 3, etc. Draw the smooth curve $A1234567M$. The second

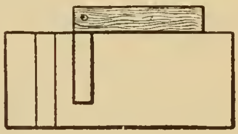


turn of the spiral is most easily obtained by marking off the distances $11'$, $22'$, $33'$, etc., each equal to AM. Construct a spiral with two turns.

8. To make a good substitute for a spirit level, take a board with one straight edge AB and draw $CD \perp AB$. From a nail F in CD hang plumb line FG. With the board vertical and edge AB on the work to be leveled, show that the plumb line must fall on DC when work is level.

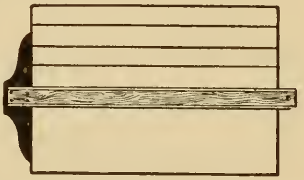


9. Draw two parallel lines. Select two points on one, and through these points draw perpendiculars to the other. Compare the lengths of the two sects between the parallel lines.



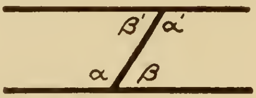
10. What principle does the carpenter use in testing the uprightness of a post or studding with a plumb line?

11. What principle is a carpenter using when he lays off parallel lines on a board by moving one arm of his square along a straight edge of the board and marking along the other?



12. What is the principle of the T-square used in the drawing course for laying off parallels?

31. Alternate angles. — When a sect connects two parallel lines, as shown in the figure, the angles β and β' (beta prime) are **alternate angles.**



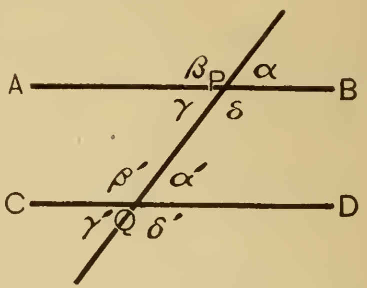
Name another pair of alternate angles. Trace β on tracing paper and apply it to β', and thus compare the two angles. Compare α and α' in the same way. Take a piece of paper whose edges are parallel and divide it by a straight cut. Apply one of the alternate angles to the other and thus compare them.

From the above measurements we may *assume* that

If a sect connects two parallel lines, the alternate angles thus formed are equal.

To THE TEACHER. — This assumption is made to avoid the early introduction of the proof by *superposition*. It may be well to have the theorem proved after the *method of superposition* has been studied.

32. Parallels cut by a transversal. — When two lines are cut by a transversal (a third line cutting the other two), eight angles are formed. They are named as follows:



α' and γ (in the figure) are *alternate interior* angles. Name another pair of alternate interior angles.

α and γ' are *alternate exterior* angles. Name another pair.

α and α' are *corresponding* angles. Name three other pairs.

α' and δ are *consecutive interior* angles. Name another pair.

α and δ' are *consecutive exterior* angles. Name another pair.

Suppose $AB \parallel CD$. What angle at P is equal to α? Why? What angle at Q is equal to γ'? Why? What angle at Q is equal to γ? Why? α and γ' are equal to what kind of angles? What axiom, then, applies to these angles? What, then, can you say of α and γ'?

Similarly, by using the alternate angles δ and β', show that $\beta = \delta'$.

From the answers to the above we have shown that

If two parallel lines are cut by a transversal, the alternate exterior angles are equal.

33. A theorem. — A **theorem** is the statement of a truth that may be deduced from other statements which have been accepted as true.

Thus, the statement of § 32, that "If two parallel lines are cut by a transversal, the alternate exterior angles are equal," is a *theorem*. This was deduced through a course of reasoning from other statements accepted as true.

In the work preceding § 32 we have reached a number of conclusions through measurements, constructions, paper folding and cutting. But most of the conclusions of geometry may be reached through a course of pure reasoning. From now on, unless otherwise indicated, the student will be expected to adhere to the latter method.

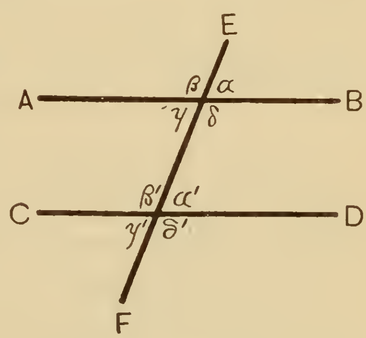
34. The hypothesis and conclusion. — A theorem consists of two parts: the **hypothesis** and the **conclusion.** The hypothesis is a statement of that which is taken as true in order to reason to the conclusion, and the conclusion is that which follows from the hypothesis.

Thus, in § 33, the *hypothesis* is that two lines are parallel and that they are cut by a third line, forming alternate exterior angles.

The *conclusion* is that the alternate exterior angles are equal.

35. The proof or demonstration. — The course of reasoning through which the conclusion is shown to follow from the hypothesis is called the **proof** or **demonstration.** The following is an illustration of a proof:

Theorem. — *If two parallel lines are cut by a transversal, the corresponding angles are equal.*



Hypothesis. $AB \parallel CD$, and EF forms the corresponding angles α and α'.

Conclusion. $\alpha = \alpha'$.

Proof. 1. $\alpha = \gamma$, because the vertical angles formed by two intersecting lines are equal.

2. $\alpha' = \gamma$, because alternate interior angles of parallel lines cut by a transversal are equal. (See assumption, § 31.)

3. Therefore $\alpha = \alpha'$, by Axiom I.

36. Theorem. — *If two parallel lines are cut by a transversal, the consecutive interior angles are supplementary.*

Hypothesis. In the figure of § 35, $AB \parallel CD$, and EF forms the consecutive interior angles δ and α'.

Conclusion. $\delta + \alpha' =$ a straight angle.

Proof. 1. $\delta + \gamma =$ a straight angle, because their exterior sides form a straight line.

2. $\alpha' = \gamma$, because alternate interior angles of parallel lines are equal.

3. Therefore $\delta + \alpha' =$ a straight angle, substituting α' for γ in step 1.

37. Theorem. — *If two parallel lines are cut by a transversal, the consecutive exterior angles are supplementary.*

Hypothesis. In the figure of § 35, $AB \parallel CD$, and EF forms the consecutive exterior angles α and δ'.

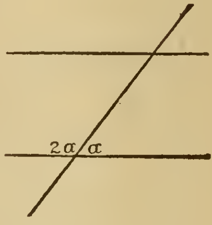
Conclusion. $\alpha + \delta' =$ a straight angle.

Proof. 1. $\alpha' + \delta' =$ a straight angle, because their exterior sides form a straight line.

2. $\alpha = \alpha'$, because corresponding angles of parallel lines are equal.

3. Therefore $\alpha + \delta' =$ a straight angle, substituting α for α' in step 1.

EXERCISES



1. α and 2α represent a pair of adjacent angles which are formed by a transversal cutting two parallel lines. Make a figure and mark (in degrees) the size of each of the eight angles.

2. Two parallel lines are cut by a transversal so that α and β are adjacent angles and α and 2β are corresponding angles. Show in a figure the size of each of the eight angles.

3. Two parallel lines are cut by a transversal making the ratio of two adjacent angles 2:3. Find the number of degrees in each of the eight angles.

4. Prove the theorem in § 35 by using the corresponding angles β and β', γ and γ', δ and δ'.

5. Prove the theorem in § 36 by using angles γ and β' as the consecutive interior angles.

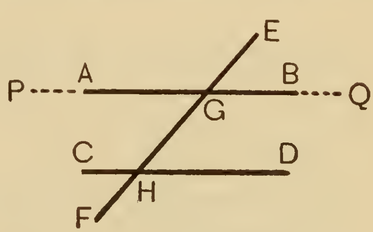
6. Prove the theorem in § 37 by using angles β and γ' as the consecutive exterior angles.

7. Prove that if a line is perpendicular to one of two parallel lines, it is perpendicular to the other also.

38. Converse theorems. — When the *hypothesis* and *conclusion* of one theorem are respectively the *conclusion* and *hypothesis* of another, one is the **converse** of the other. Thus, the converse of the theorem in § 35 is: *If two straight lines are cut by a transversal making the corresponding angles equal, the lines are parallel.* The converse of a theorem is not necessarily true. Thus, the converse of the statement "All right angles are equal" is, "All equal angles are right angles," which is *not* true.

In the following sections the converses of the preceding theorems on parallel lines are proved.

39. Theorem. — *If two straight lines are cut by a transversal making the alternate interior angles equal, the lines are parallel.*



Hypothesis. AB and CD are cut by EF at G and H, respectively, making $\angle AGH = \angle DHG$.

Conclusion. $AB \parallel CD$.

Proof. 1. Suppose PQ drawn through G parallel to CD.

2. Then $\angle PGH = \angle DHG$, by § 31.

3. But $\angle AGH = \angle DHG$, by hypothesis.

4. Therefore $\angle PGH = \angle AGH$, by Axiom I.

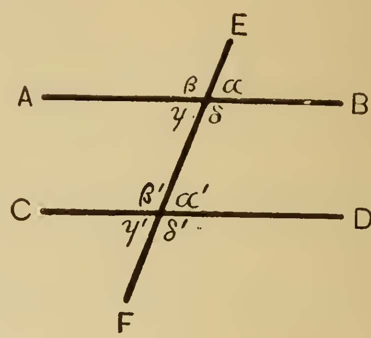
5. Hence, PQ and AB must coincide.

6. Therefore, since PQ is parallel to CD, AB, which coincides with PQ, is parallel to CD.

Note. — The line PQ drawn to aid in the proof of this theorem is called an **auxiliary line**. Such lines, not given in the figure, are used frequently in the proofs of theorems. They are denoted by dotted lines.

40. Theorem. — *If two straight lines are cut by a transversal making the alternate exterior angles equal, the lines are parallel.*

Hypothesis. AB and CD are cut by EF, making $\alpha = \gamma'$.



Conclusion. $AB \parallel CD$.

Proof. 1. $\gamma = \alpha$. Why?

2. $\alpha' = \gamma'$. Why?

3. But $\alpha = \gamma'$, by hypothesis.

4. $\therefore \gamma = \alpha'$, by Axiom I.

5. $\therefore AB \parallel CD$, by § 39.

41. Theorem. — *If two straight lines are cut by a transversal making the corresponding angles equal, the lines are parallel.*

Hypothesis. In the figure of § 40, $\alpha = \alpha'$.

Conclusion. $AB \parallel CD$.

Suggestions. Compare γ and α, then γ and α'. Hence show that $AB \parallel CD$, by § 39. Write out complete proof.

42. Theorem. — *If two straight lines are cut by a transversal making the consecutive interior angles supplementary, the lines are parallel.*

Hypothesis. In the figure of § 40, $\delta + \alpha' =$ st. $\angle$.

Conclusion. $AB \parallel CD$.

Suggestions. Show that $\gamma = \alpha'$, then apply § 39.

43. Theorem. — *If two straight lines are cut by a transversal making the consecutive exterior angles supplementary, the lines are parallel.*

Hypothesis. In the figure of § 40, $\alpha + \delta' =$ st. $\angle$.

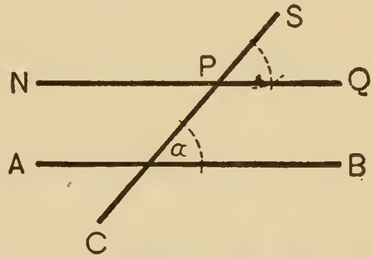
Conclusion. $AB \parallel CD$.

Suggestion. Proceed as in § 42.

EXERCISES

1. Prove the theorem in § 39 when $\angle HGB = \angle GHC$.
2. Prove the theorem in § 40 when $\beta = \delta'$.
3. Prove the theorem in § 41 when $\beta = \beta'$; when $\gamma = \gamma'$; when $\delta = \delta'$.
4. Prove the theorem in § 42 when γ and β' are supplementary.
5. Prove the theorem in § 43 when β and γ' are supplementary.

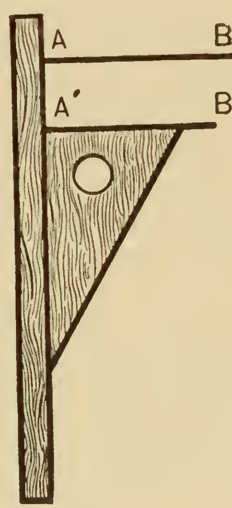
44. To draw parallels. — The theorem of § 41 suggests another method of drawing one line parallel to another. The method is as follows: Suppose the line is to pass through point P and be parallel to AB. Draw any line through P cutting line AB, making with AB an angle α. Now on the side PS construct the corresponding angle QPS equal to α. Then line PQ is parallel to AB.



What other theorems suggest methods of drawing a line parallel to a given line?

Assumption. — *Through a given point only one line can be drawn parallel to a given line.*

45. To use a triangle. — Take a *triangle* (also called a *set-square*) and rule a straight line AB along one of its edges. Holding the triangle firmly, bring your straightedge against one of the other edges. Now holding the straightedge firmly, slide the triangle along the straightedge to another position and rule a second line $A'B'$. The two lines AB and $A'B'$ will be parallel. Upon what theorem or theorems does this depend?



EXERCISES

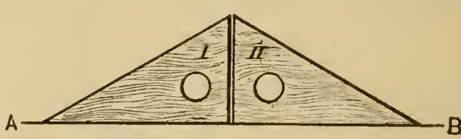
1. Draw three pairs of parallel lines, using successively each of the three sides of your triangle against the ruler.

2. Practice drawing parallel lines with compasses and ruler until you can do it accurately. Test your work by drawing any transversal and measuring the angles that should be equal. Which is more likely to be in error, your test or your drawing? Test your work with a triangle.

3. By the method of § 44, draw a line through a given point and parallel to a given line.

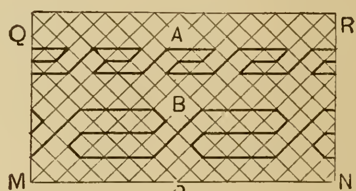
4. Using the triangle as in § 45, draw a line through a given point and parallel to a given line.

5. Test the accuracy of the right angle of your triangle by erecting a perpendicular to a line AB, using position I as in the figure, then turning the triangle over to position II to see if the perpendicular coincides with the edge of the triangle in its new position. Upon what principle does this test depend?

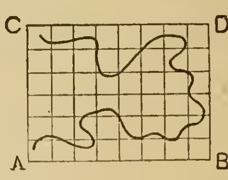


6. The "square network" shown in the figure is used in designing for drawing a great variety of patterns. The patterns A and B drawn upon it are examples of Arabian frets. The best way to rule the square network is to draw a horizontal line MN, and mark off equal divisions on it. At each point of division, by use of the triangle, draw two lines, such as PQ and PR, each making 45° with MN.

Draw such a network, then upon it construct a pattern, either an original design of your own or a copy of these Arabian frets.



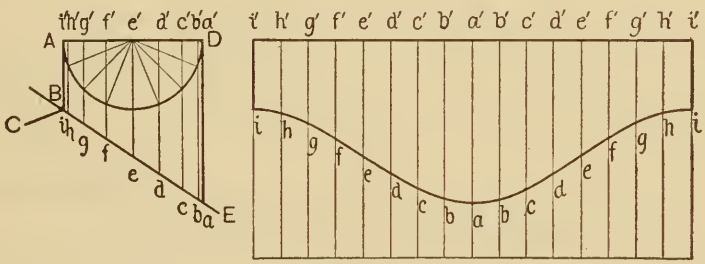
7.* An enlarged or reduced copy of a picture, map, or plan may be drawn as follows: On the given map, say, draw a sect AB, and divide it into a number of equal parts. At A draw AC perpendicular to AB, dividing it into parts equal to those of AB. Through the points of division of AB draw parallels to AC, and



through the points of division of AC draw parallels to AB, forming a network of squares, as shown. To make a copy of the figure twice its present size, say, draw a sect MN twice as long as AB, and divide it into as many equal parts as AB. Then upon MN construct a network of squares by the same method as in the given map. The points of the new map corresponding to the various points of the given map may then be marked on the corresponding squares, and joined by a smooth curve.

Construct a copy of this map twice it present size. Use compasses, straightedge, and triangle.

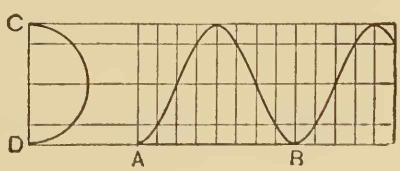
8.* To draw the pattern for cutting out sheet metal in the construction of elbows with any given angle ABC and any given diameter AD: Draw BE to bisect angle ABC. Draw AD equal to the given diameter



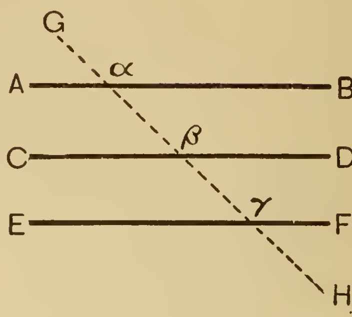
and perpendicular to AB. Draw DE parallel to AB. Bisect AD at e', and with AD as diameter draw a half circle. Divide the straight angle at e' into eight equal angles by rays meeting the circle in points. Through these points draw parallels to AB, as in the figure, meeting BE at b, c, etc. Now draw $i'i'$ equal to $3\frac{1}{7}$ times AD. Divide $i'i'$ into sixteen equal parts. Through the points of division thus obtained draw perpendiculars to $i'i'$, and on these measure off distances equal to ii', hh', gg', etc., as in the figure. Through the ends of these lines draw a smooth curve. The metal must be cut out along this curve.

Draw such a pattern for an elbow with any given angle.

9.* Draw the helix or thread of a screw of given pitch AB (the distance between two consecutive threads) and diameter CD. The figure suggests the construction.



46. Theorem. — *Two straight lines parallel to the same straight line are parallel to each other.*

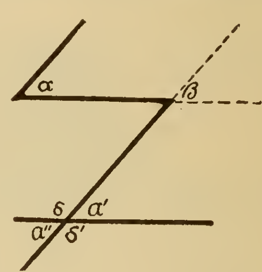


Hypothesis. $AB \parallel EF$ and $CD \parallel EF$.

Conclusion. AB CD.

Suggestion. Draw a transversal, forming the corresponding angles α, β, and γ. Show that $\alpha = \beta$.

47. Theorem. — *Angles having their sides parallel, each to each, and extending in the same direction, or in opposite directions from the vertex, are equal. If one pair of parallel sides extend in the same direction and the other pair in opposite directions from the vertex, the angles are supplementary.*



NOTE. — It will be observed that in the following proof no reasons are stated for the conclusions in the various steps. It is expected that the student will furnish the reasons. *In many of the following proofs the reasons for the various steps are omitted from the book. But the student is expected to give the reason for every step in the proof.* **In no case is a demonstration complete without the reason being given for every step in it.**

Hypothesis. (1) α and α' have their sides parallel and extending in the same direction.

(2) α and α'' have their sides parallel and extending in opposite directions.

(3) α and δ, and α and δ' have one pair of parallel sides in the same direction and the other pair in opposite directions.

Conclusion. $\alpha = \alpha'$, $\alpha = \alpha''$, $\alpha + \delta =$ a st. $\angle$ and $\alpha + \delta' =$ a st. $\angle$.

Proof. 1. If the non-parallel sides do not intersect, produce them until they do, forming angle β.

2. $\alpha = \beta = \alpha' = \alpha''$.

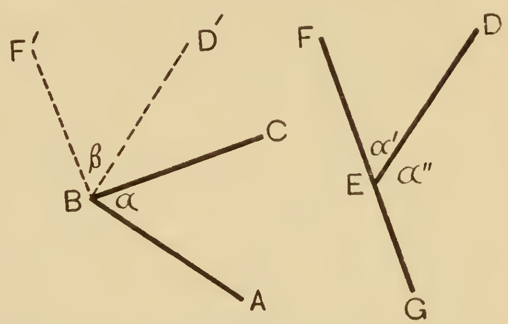
3. Also $\alpha' + \delta = \text{a st.} \angle$.

4. $\therefore \alpha + \delta = \text{a st.} \angle$.

5. And $\delta = \delta'$.

6. $\therefore \alpha + \delta' = \text{a st.} \angle$.

48. Theorem. — *Angles having their sides perpendicular, each to each, and both acute or both obtuse, are equal. If one is acute and the other obtuse, they are supplementary.*



Hypothesis. $\angle s\ \alpha$ and α' are acute and α'' is obtuse. Also side $BC \perp FG$ and $BA \perp ED$.

Conclusion. $\alpha = \alpha'$, and α and α'' are supplementary, *i.e.* $\alpha + \alpha'' = \text{a st.} \angle$.

Proof. 1. Draw $BF' \parallel EF$ and $BD' \parallel ED$ and extending in the same directions.

2. Then $\beta = \alpha'$.

3. Also $BF' \perp BC$ and $BD' \perp BA$.

4. Then $\beta = \alpha$.

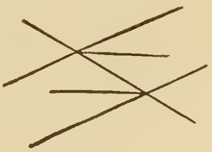
5. $\therefore \alpha = \alpha'$.

6. Also $\alpha' + \alpha'' = \text{a st.} \angle$.

7. $\therefore \alpha + \alpha'' = \text{a st.} \angle$. (Substituting α for α' in step 6.)

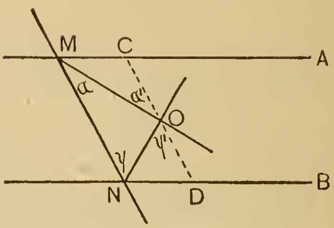
EXERCISES

1. Two parallel lines are cut by a transversal making one interior angle 35°. Find the value of each of the other seven angles.



2. The bisectors of the alternate interior angles formed by two parallels cut by a transversal are parallel.

3. What can you say of the bisectors of the corresponding angles in Exercise 2? Prove the answer.



4. The bisectors of two consecutive interior angles are perpendicular to each other.

Hypothesis. — Parallels AM and BN are cut by MN, and MO and NO are the bisectors of consecutive interior angles, meeting at O.

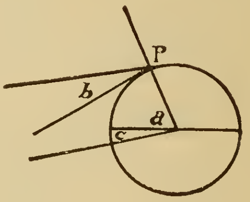
Conclusion. — $MO \perp NO$, *i.e.* $\angle MON =$ a rt. $\angle$.

Suggestions. — 1. Through O draw $CD \parallel MN$.

2. $\because \angle COD$ is a st. $\angle$,

3. $\therefore \angle MON$ can be proved a rt. $\angle$ by proving $\alpha + \gamma =$ a rt. $\angle$;

4. and $\alpha + \gamma$ can be proved equal to a rt. $\angle$ by § 36.

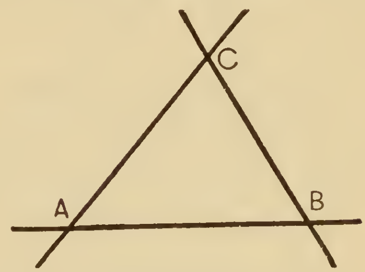


5.* The latitude of a place P on the earth's surface, such as that of a ship at sea, is obtained by observing the altitude of the sun (angle above the horizon) at noon. Prove that the latitude of P equals the difference between the zenith distance of the sun (90° minus altitude of sun) and the declination of the sun (angle it makes with the plane of the equator); *i.e.* $a = (90 - b) - c$.

CHAPTER III

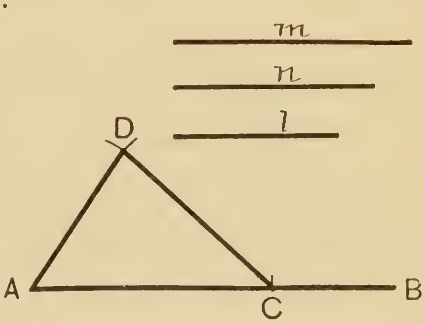
TRIANGLES: INDIRECT PROOF

49. Triangles. — Three straight lines, not meeting in the same point, will inclose a portion of a plane, as shown in the figure. A portion of the plane bounded by AB, BC, and CA is a **triangle**. The sects AB, BC, and CA are the **sides**, and the points where they meet are the **vertices**.



The triangle is named by its three vertices. Thus, triangle ABC. In writing, the word "triangle" is often replaced by the symbol $\triangle$. Thus, triangle ABC is written $\triangle ABC$.

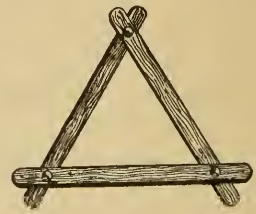
50. Construction. — *A triangle may be constructed with its sides equal in length to given sects.*



The work is as follows: Let the given sects be m, n, and l. On a straight line AB cut off AC equal to m. Adjust the legs of the compasses to the ends of n. Place one leg at C, and with the other describe an arc. Now adjust the compasses to the ends of l. Place one leg of the compasses at A and with the other draw an arc cutting the first arc at some point. Call this point D. Draw AD and CD. Now $\triangle ACD$ is evidently the triangle desired.

If sect m had been longer than the sum of the other two sects, would it have been possible to draw the triangle? Why?

If $m = n + l$, could the triangle have been drawn? Why? Can you think of any other cases in which the triangle could not have been drawn? Make other triangles with the three sides given in this section. If they are each cut out carefully, or traced upon tracing paper, will any one exactly fit upon any other?



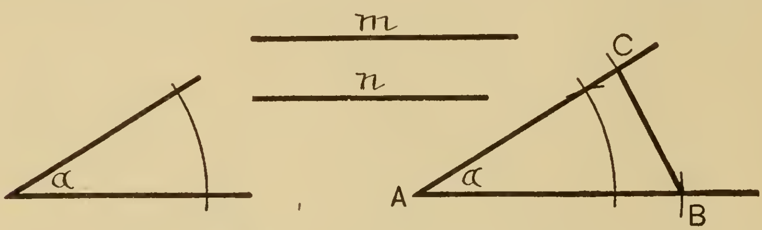
Make a triangular framework from three sticks nailed together with but one nail at each joint, as in the figure. Can you alter the shape of the framework without breaking a stick or tearing one of the joints apart?

It seems evident from our constructions, our tracing and application of one triangle to the other, and our experiment with the stick framework, that

A triangle is fully determined when its three sides are given.

51. Construction. — *A triangle may be constructed when two of its sides and their included angle are given.*

The contruction is as follows: Construct an angle equal to



the given angle α. From the vertex of this angle lay off on the sides sects AB and AC equal to the given sides m and n, respectively. Join the extremities of the sects. Show all the work in doing this.

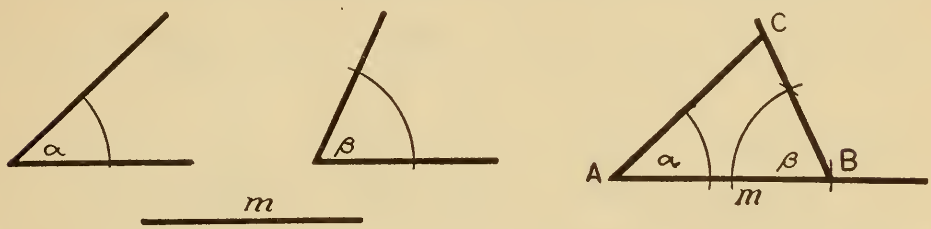
If you should construct other triangles having these sides and this angle, would they exactly fit, one upon another, if cut out and applied? Test your answer by constructing two triangles from the same data and applying one to the other either by tracing one and placing the tracing upon the other, or by cutting one out and placing it upon the other.

The answers to the above questions lead us to the conclusion that,

A triangle is fully determined by two sides and their included angle.

52. Construction. — *A triangle may be constructed when a side and the two adjacent angles are given.*

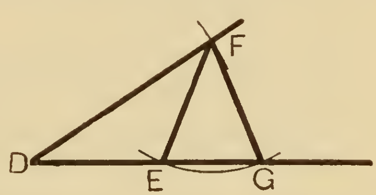
The construction is as follows: On a straight line lay off



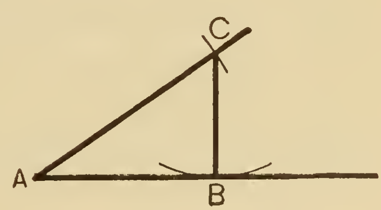
a sect AB equal to the given side m. At one end of the sect draw an angle equal to one of the given angles, and at the other end draw an angle equal to the other given angle, as shown. Produce the sides of these angles until they intersect at a point C. Then $\triangle ABC$ is the required triangle.

Draw two triangles having a side and two adjacent angles of one equal to a side and two adjacent angles of the other. Apply one to the other, and state your conclusion.

53. Construction. — If we were given two sides of a triangle and the angle opposite the shorter of them, we could construct two triangles (as DEF and DGF in the figure) except when the shorter side just equaled the distance to the third side, or was shorter than this distance.



Draw several triangles from such data. We conclude, then, that *a triangle is not determined by two sides and the angle opposite the shorter side.*



Is the triangle determined by two sides and the angle opposite the larger side? Make constructions from such data.

54. Classification of triangles as to sides. — Triangles are classified, according to the lengths of their *sides*, as follows:

A **scalene** triangle has *no two* of its sides equal.

An **isosceles** triangle has *two* of its sides equal.

An **equilateral** triangle has *all* of its sides equal.

SCALENE ISOSCELES EQUILATERAL

EXERCISES

1. Draw an isosceles triangle with each of its equal sides equal to a given sect l and the third side equal to a given sect m.

2. Draw an isosceles triangle each of whose equal sides is twice the third side.

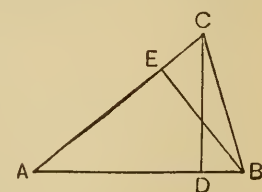
3. Can you draw an isosceles triangle whose equal sides are each equal to or less than half the third side?

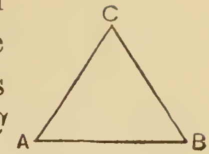
4. Draw an equilateral triangle with its sides equal to a given sect l.

5. Construct two isosceles triangles upon the same third side, one above it and one below it.

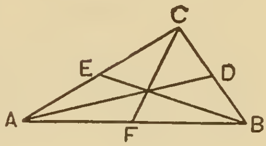
6. Draw two equilateral triangles upon the same side, one above it and the other below it.

55. The special lines of a triangle. — Any side of a triangle may be called the **base**. Then its **altitude** is the perpendicular distance from the vertex opposite the base to the base. Thus, a triangle may have three distinct bases and three distinct altitudes. In the figure, if AC is considered the base, BE is the altitude.





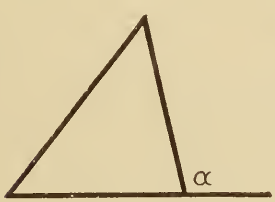
56. In the **isosceles triangle** the intersection of the two equal sides is called **the vertex** of the triangle, and the side opposite the vertex is called **the base**. In the figure, if $AC = CB$, C is called the *vertex* and AB the *base*.



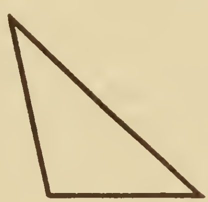
57. The medians of a triangle are the sects drawn from the vertices to the middle points of the opposite sides. Thus, in the figure, AD, BE, and CF are medians.

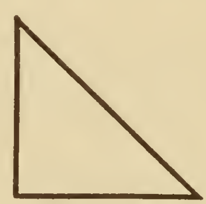
58. Classification of triangles as to angles. — Triangles are classified, according to their *angles*, as follows:

A triangle *all* of whose angles are acute is an **acute** triangle.
A triangle *one* of whose angles is obtuse is an **obtuse** triangle.
A triangle *one* of whose angles is right is a **right** triangle.
A triangle *all* of whose angles are equal is **equiangular**.



ACUTE


OBTUSE


RIGHT

In any triangle, an angle formed by a side and a side produced, as α in the above figure, is called an **exterior angle** of the triangle.

In the *right triangle*, the side opposite the right angle is the **hypotenuse**, and the other two sides are the **legs**.

EXERCISES

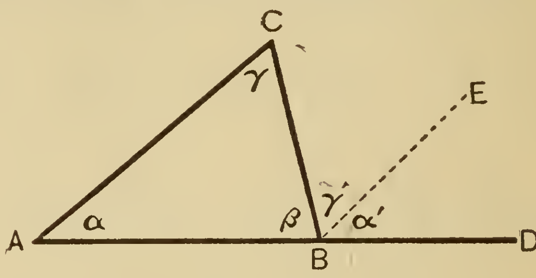
1. Draw a triangle such that the foot of the altitude is on the base produced. What kind of triangle is it?

2. Draw a triangle whose altitude coincides with one of the sides. What kind of triangle is it?

3. Draw a triangle whose altitude falls within the triangle. What kind of angles has it at the base?

4. Draw a triangle each of whose altitudes falls within the triangle. What kind of triangle is it?

59. Theorem. — *The sum of all the angles of any triangle is equal to a straight angle.*



Hypothesis. The figure ABC is a triangle whose angles are α, β, γ.

Conclusion. $\alpha + \beta + \gamma$ = a st.∠.

Proof. 1. Draw $BE \parallel AC$, and produce AB to D.

2. Then $\gamma = \gamma'$ and $\alpha = \alpha'$. (?)

3. And $\alpha' + \gamma' + \beta$ = a st.∠. (?)

4. ∴ $\alpha + \beta + \gamma$ = a st.∠. (?)

Note. — This very important theorem is attributed to Pythagoras, a Greek philosopher and mathematician, born about 569 B.C. But there is reason for believing that Thales, a Greek born about 640 B.C., knew that it is true. The demonstration given here is attributed to Euclid, the Greek mathematician, born about 300 B.C., who wrote the *Elements*, the great treatise on elementary geometry.

60. A corollary. — A **corollary** is a theorem that is deduced immediately from a given theorem or definition. The student is expected in the case of every corollary to supply the proof of it. The following corollaries follow from the preceding theorem:

61. Corollary 1. — *If the values of two angles of a triangle are known, the value of the third angle is known.*

62. Corollary 2. — *If the value of an acute angle of a right triangle is known, the value of the other acute angle is also known.*

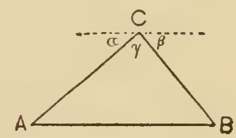
63. Corollary 3.—*In any triangle there must be at least two acute angles.*

64. Corollary 4.—*An exterior angle of any triangle is equal to the sum of the two opposite interior angles, and hence greater than either one of them.*

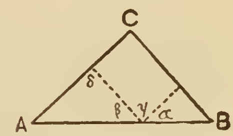
EXERCISES

Note. — Observe that all auxiliary lines are so drawn as to make the proof depend upon some known theorem or definition.

1. Prove the theorem in § 59 by drawing a straight line through the vertex parallel to the base.



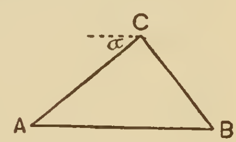
2. Prove the same theorem by taking any point on the base and through it drawing straight lines parallel to the other two sides.



Suggestions. — 1. ∵ it is known that $\alpha + \beta + \gamma$ = a st.∠,

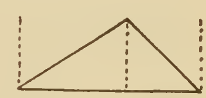
2. this suggests that we prove $\angle A + \angle B + \angle C = \alpha + \beta + \gamma$.

3. Compare α and $\angle A$, and $\angle B$ and β, also $\angle C$ and γ.

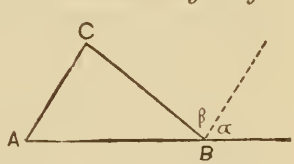


3. Prove the same theorem by drawing a ray *from* the vertex parallel to the base.

4. Prove the same theorem by drawing perpendiculars from the extremities of the base and from the vertex to the base.



5. Prove, independently of theorem § 59, that *an exterior angle of a triangle is equal to the sum of the two opposite interior angles.*



Suggestion. — Compare α with $\angle A$, and β with $\angle C$.

6. Prove the theorem of § 59 by using Exercise 5.

7. I may test my accuracy as follows in measuring angles with an instrument: I drive stakes in the ground at three points, A, B and C, so that they are not in a straight line. I set the instrument over A and measure the angle between the directions to B and C. Then I set it over B and

measure the angle between the directions to C and A. Then I set it over C and measure the angle between the directions to A and B. The angles are found to be $46° 12'$, $32° 25'$, and $101° 53'$. Am I accurate?

8. If one angle of a triangle is $37° 25'$, and the other angles are equal, find the value of each.

9. If the three angles of a triangle are equal, how many degrees in each?

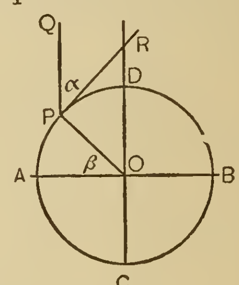
10. Of the three angles of a triangle the second is twice the first and the third three times the first. How many degrees in each?

SUGGESTION. — Form an equation.

11. One of the acute angles of a right triangle is four times the other. How many degrees in each?

12. In a certain triangle an exterior angle is twice the adjacent interior angle, and the two opposite interior angles are equal. How many degrees in each of the angles of the triangle?

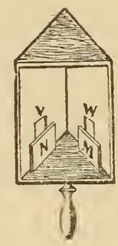
13.* Sailors find their latitude at sea by observing the altitude of the North Star. Prove that the latitude of the observer equals the altitude (angular distance above the horizon) of the North Star.



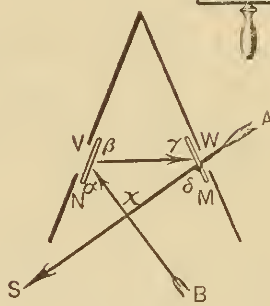
SUGGESTION. — CD is the axis of the earth which points toward the North Star, AB the equator at right angles to the axis, PQ the direction of the star from the point of observation P, PO the vertical line through P, and PR the horizontal line at right angles to vertical. On account of the great distance to the star the direction of the star from the observer is considered parallel to the axis of the earth. Prove $\alpha = \beta$.

14.* In forestry, when it is necessary to lay off right angles, this is done sometimes by means of an instrument called an "optical square." It consists of a small metal box with a triangular bottom and with but two side walls. There are openings, or windows, W and V, in the side walls, and below these two vertical mirrors M and N are set at an angle of $45°$ with each other. The observer at S looks directly into the box through the open side, and holds it so that an object A can be seen through the window W. At the same time the image of an object B is

seen in the mirror M below W, and in line with A. The principle is that B is imaged at N, then this reflected to M, then back to the eye at S. Show that the directions to A and to B are at right angles.



SUGGESTION. — Show that angle $x = 90°$. By a law of light, a ray of light is reflected from a surface at an angle equal to the angle at which it strikes it, *i.e.* $\alpha = \beta$ and $\gamma = \delta$.



65. Remark. — It has been seen in the above exercises that there may be many ways of attacking a theorem. It also was seen from Exercises 5 and 6 that sometimes a corollary to a theorem may be proved without the aid of the theorem, and the theorem then made its corollary; for Exercise 5 was a corollary to the theorem of § 59, while Exercise 6 was proved directly by Exercise 5. Hence Cor. 4, § 64, could as well have been made the theorem, and the theorem the corollary. It will be noticed, then, from this, that the proof of a theorem depends only upon what has already been known. It is seen also that the *auxiliary* lines were so drawn as to convert the given figure into one whose properties were known from some preceding theorem.

66. Congruent figures. — When one figure can be applied to another so that the two figures may be made to fit or coincide throughout all their parts, the figures are **congruent**. It is evident that in congruent figures any two corresponding parts are equal, since they can be made to coincide. Congruent figures, then, are *equal*, but equal figures are not necessarily congruent. That is, one may have the same area as the other, although they do not have the same shape.

In written work the words "is congruent to" are often expressed by the symbol $\cong$.

67. Proof by superposition. — Instead of actually tracing one figure and placing it upon another, or cutting one out and placing it upon the other, one may be *imagined* to be placed upon the other. The proof of many of the fundamental theorems depends upon such a method, called **the method of superposition.** In using this method we should keep in mind that:

(1) *One straight line can* ALWAYS *be placed upon another.*

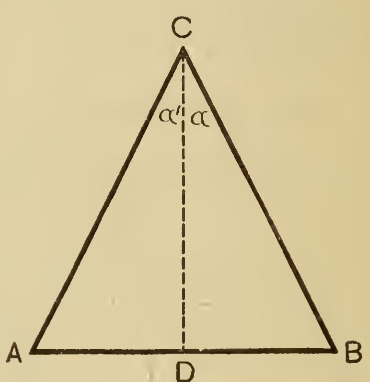
(2) *If two angles are equal, they may be placed so that their vertices coincide and the sides of the one fall upon the sides of the other.*

(3) *If two angles are equal, and are placed one upon the other, and the vertex and a side of one fall upon the vertex and side of the other, the remaining sides* MUST *coincide.*

(4) Also, to apply one figure to another, *one may be supposed to be moved about in the plane in any manner, or to be revolved about any point or line, without distortion, until the figure falls upon the adjacent portions of the plane.*

The following theorem illustrates the method.

68. Theorem. — *In any isosceles triangle, the angles opposite the equal sides are equal.*



Hypothesis. $\triangle ABC$ is isosceles, with $AC = BC$.

Conclusion. $\angle A = \angle B$.

Proof. 1. Draw CD bisecting $\angle C$ and cutting AB in D.

2. Apply $\triangle DBC$ to $\triangle ADC$, by folding it about CD as axis, until it falls in the plane of ADC.

3. $\alpha = \alpha'$. (?)

4. $\therefore$ CB will fall along CA.

5. And $CB = CA$. (?)

6. $\therefore B$ will fall upon A. (?)

7. $\therefore DB$ will fall upon DA, and $\angle B$ and $\angle A$ will coincide throughout.

8. $\therefore \angle A = \angle B$, for they are congruent, § 66. (?)

69. Corollary 1. — *An equilateral triangle is also equiangular.*

70. Corollary 2. — *Each angle of an equilateral triangle is* 60°.

71. Corollary 3. — *In any isosceles triangle the bisector of the vertical angle is the perpendicular bisector of the base.*

72. Corollary 4. — *The median from the vertex of any isosceles triangle bisects the vertical angle.*

73. Remark. — In any triangle, ABC, there are three straight lines that may be considered with reference to any vertex, and one with reference to the corresponding base.

(1) *The bisector of the angle* A.

(2) *The altitude from* A.

(3) *The median from* A.

(4) *The perpendicular bisector of* BC.

In general, these are four distinct lines, but we have seen from the above theorem and its corollaries that if the triangle is isosceles and A is the vertical angle, they are all one and the same straight line.

EXERCISES

1. If the angle at the vertex of an isosceles triangle is 60°, each of the other angles is 60°.

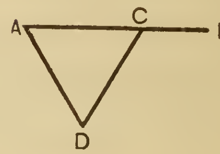
2. One of the angles at the base of an isosceles triangle is 45°. Find the size of each of the other angles.

3. Construct an angle of 60° by use of the compasses and straight-edge.

4. Construct an angle of 30° by use of the compasses and straight edge.

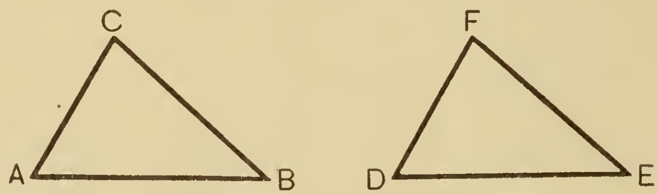
5. Trisect a right angle by use of the compasses and straightedge.

6. A surveyor often needs to lay off an angle of 60° with a given line. If AB is the line, and A the point on it, show that he may do this as follows: With one end of the 50-ft. tape held at A and the other at C, which is 25 ft. from A, let a third person take the tape at the 25-ft. mark, and stretch both parts straight, locating a point D. Then $\angle DAC = 60°$.



74. The congruence of triangles. — The constructions of §§ 50, 51, and 52 would lead us to infer that triangles are congruent when certain angles or sides are equal. We shall now establish, by the method of superposition, some very important theorems about the congruence of triangles.

75. Theorem. — *If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another, the triangles are congruent.*



Hypothesis. $\triangle ABC$ and DEF have $AB = DE$, $AC = DF$, and $\angle A = \angle D$.

Conclusion. $\triangle ABC \cong \triangle DEF$.

Proof. 1. Place $\triangle DEF$ upon $\triangle ABC$ so that DE coincides with its equal AB, D falling on A and E on B.

2. Then $\because \angle D = \angle A$, DF *must* fall along AC.

3. And $\because DF = AC$, F must fall on C.

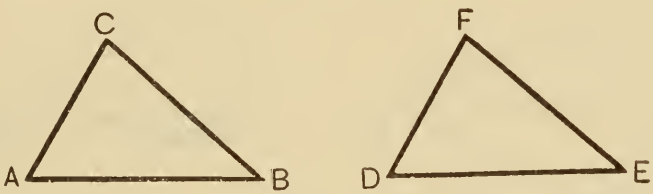
4. $\therefore EF$ coincides with BC. (Why?)

5. $\therefore$ the $\triangle$ coincide throughout and are congruent by the definition of congruence.

76. Corollary 1. — *Two right triangles are congruent if the legs of one are equal respectively to the legs of the other.*

77. Corollary 2. — *In congruent triangles corresponding altitudes are equal.* [When $\triangle ABC$ and DEF (§ 75) were superposed, F and C coincided. ∴ perpendiculars from C to AB and from F to DE must coincide. Why?]

78. Theorem. — *If two angles and a side of one triangle are equal respectively to two angles and the corresponding side of another, the triangles are congruent.*



NOTE. — It does not matter what two pairs of angles are given equal, since, if two pairs are equal, the other angles are equal. Why?

Hypothesis. $\triangle ABC$ and DEF have $\angle A = \angle D$, $\angle B = \angle E$ and $AB = DE$.

Conclusion. $\triangle ABC \cong \triangle DEF$.

Suggestions. 1. Place $\triangle ABC$ upon $\triangle DEF$ so that AB will coincide with DE, and A falls upon D and B upon E. (Why can you do this?)

2. What direction will AC take with respect to DF? (Why?)

3. Upon what line of $\triangle DEF$ must C lie?

4. What direction must BC take?

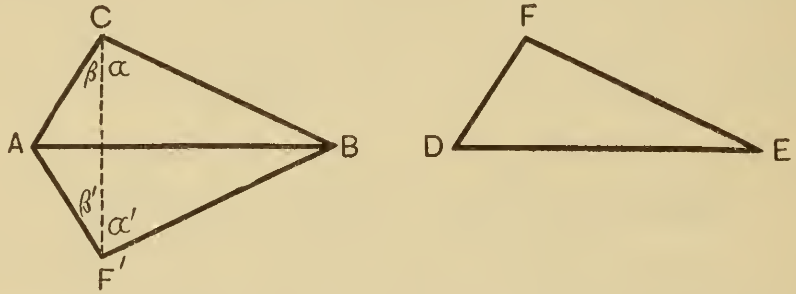
5. Upon what other line of $\triangle DEF$ must C then lie?

6. Show why C must then coincide with F and that the triangles are congruent.

79. Corollary 1. — *Two right triangles are congruent if the hypotenuse and an acute angle of one are equal to the hypotenuse and an acute angle of the other.*

80. Corollary 2. — *Two right triangles are congruent if a leg and an acute angle of one are equal to the corresponding leg and acute angle of the other.*

81. Theorem. — *If three sides of one triangle are equal respectively to three sides of another, the two triangles are congruent.*



Hypothesis. $\triangle$s ABC and DEF have $AB = DE$, $BC = EF$, and $AC = DF$.

Conclusion. $\triangle ABC \cong \triangle DEF$.

Proof. 1. Apply $\triangle DEF$ to $\triangle ABC$ so that DE will coincide with its equal AB, D falling on A and E on B, but F at F' opposite C.

2. Draw CF'.

3. $\triangle$s $CF'B$ and CAF' are isosceles.

4. $\therefore \alpha = \alpha'$ and $\beta = \beta'$.

5. $\therefore \alpha + \beta = \alpha' + \beta'$, *i.e.* $\angle ACB = \angle AF'B$.

6. $\therefore \triangle ABC \cong \triangle AF'B$. (Why?)

7. But $AF'B$ is only another position of DEF, and $\therefore \triangle ABC \cong \triangle DEF$.

Let the student draw a figure and prove the theorem, when $\angle$s B and E are right; when obtuse.

Note. — Notice that the method of superposition could not be used in the above theorem, for the angles not being known, we did not know where sides DF and EF would fall when DE coincided with AB.

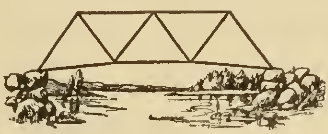
EXERCISES

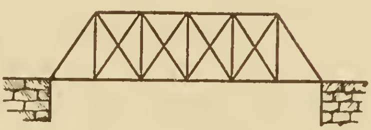
1. Why is a crane completely supported by a single diagonal brace?

Note. — Exercises 1, 2, 3, and 4 are applications of § 50.

2. Show why a two-sided roof is braced sufficiently when one tie beam connects each pair of rafters.

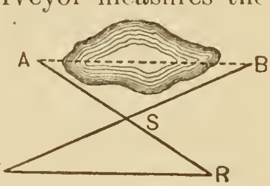
3. Some boys made a bridge across a brook. The figure shows how the girders were fastened. Show why the bridge cannot collapse.





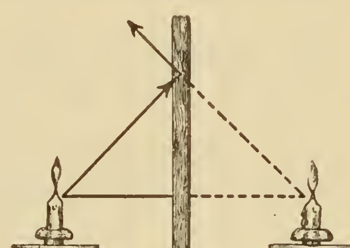
4. Why is a long span of a bridge, in which the truss is made with queen posts and diagonal rods, as represented in the diagram, sufficiently supported?

5. Justify the following method by which a surveyor measures the distance from A to the inaccessible point B. Locate a third point S from which the distances to A and B can be measured. Measure AS, and by sighting from A to S extend this line to R, so that $AS = SR$. Likewise, measure BS, and extend it to T so that $BS = ST$. Measure the distance RT. Then $AB = RT$



Suggestion. — Prove triangles congruent.

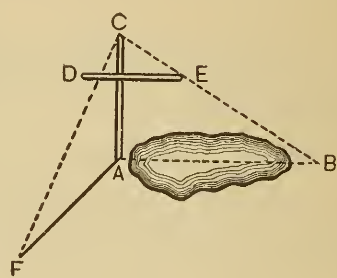
6. Every person is familiar with the fact that if an object is placed before a mirror, its image appears to be as far behind the mirror as the object is in front. Prove that this must always be so.



Note. — When a ray of light strikes a reflecting surface, the striking ray and the reflected ray make equal angles with the plane of the surface.

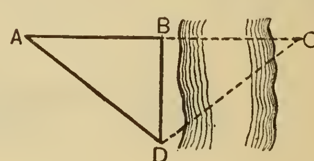
Suggestion. — Prove triangles congruent.

7. In the sixteenth century, the distance from A to the inaccessible point B was found by use of an instrument consisting of a vertical staff AC to which was attached a horizontal cross bar DE that could be moved up and down on the staff. Sighting from C to B, DE was lowered or raised until C, E, and B were in a straight line. Then the whole instrument was revolved, and the point F at which the line of sight CE struck the ground again was marked, and FA measured. Show that $FA = AB$.

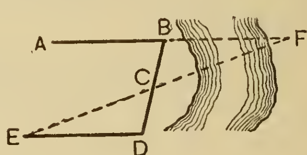


SUGGESTION. — Prove triangles congruent.

8. In surveying, the distance from B to the inaccessible point C may be found as follows: Run a line BD at right angles to BC. Prolong CB through B. From D locate a point A in the prolongation of BC so that $\angle BDA = \angle CDB$. Measure AB. Show that $BC = AB$, and hence that BC is found.



9. To measure the distance from B to the inaccessible point F, run BD in any convenient direction. Locate C, the middle point of BD. From D, with an instrument, run $DE \parallel BF$, and locate E in line with C and F. Show that $BF = ED$, and hence is determined.



10. If from a point in a perpendicular to a line two sects are drawn to the line, cutting off equal sects on the line from the foot of the perpendicular, the two sects drawn are equal.

82. Methods of proof. — The preceding exercises illustrate a method of proving sects equal or angles equal.

(1) *The equality of sects and angles is very generally proved by means of congruent triangles.*

(2) *When sects and angles that are to be proved equal are not parts of congruent triangles, try to draw auxiliary lines that will give such triangles.*

(3) *Angles are often proved equal by use of parallel lines.*

In the proofs of the following exercises, draw auxiliary lines forming triangles, then by *proving these triangles congruent*, prove that the sects or angles are equal, as the case may require.

EXERCISES

Note. — The constructions to be justified in these exercises were assumed without proof in the early part of the work. It is important that the proofs be now supplied.

1. Prove that, by the construction in § 15, $\angle POR = \angle BAC$.

2. Prove that, by the construction in § 26, $\angle ABC$ is bisected by BP.

3. Prove that, by the construction in § 11, AB is bisected at C.

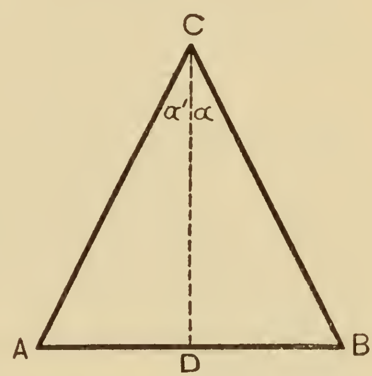
4. Prove that, by the construction in II, § 28, $DC \perp AB$.

83. Theorem. — *If two angles of a triangle are equal, the triangle is isosceles.*

Hypothesis. $\triangle ABC$ has $\angle A = \angle B$.

Conclusion. $\triangle ABC$ is isosceles, *i.e.* $AC = BC$.

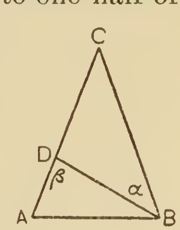
Suggestion. Draw CD *bisecting* $\angle C$. (Why?) Prove $\triangle ADC \cong \triangle DBC$. (How?)



84. Corollary. — *An equiangular triangle is also equilateral.*

EXERCISES

1. If the vertical angle of an isosceles triangle is equal to one half of a base angle, the bisector of a base angle divides the triangle into two isosceles triangles.

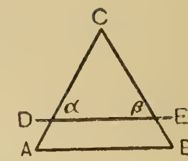


Hypothesis. — $\angle C = \frac{1}{2} \angle A$, and BD, the bisector of $\angle B$, cuts AC in D.

Conclusion. — $\triangle s$ ABD and DBC are isosceles.

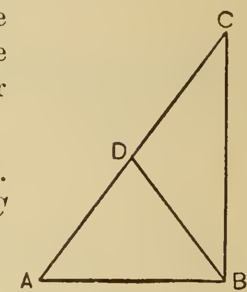
Suggestion. — Prove that $\angle C = \alpha$, and that $\beta = \angle A$.

2. A straight line drawn through the equal sides of an isosceles triangle, and parallel to the base, makes equal angles with the equal sides.



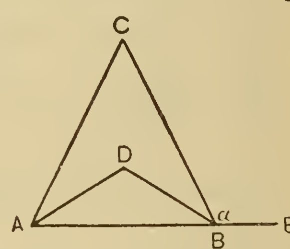
Suggestion. — Compare α with $\angle A$ and β with $\angle B$.

3. If one of the equal sides of an isosceles triangle be produced through the vertex its own length, the straight line joining its extremity with the nearer extremity of the base is perpendicular to the base.



Suggestions. — If $CB \perp AB$, then $\angle ABC$ is a rt. $\angle$. This suggests that $\angle ABC = \angle A + \angle C$, for $\angle ABC + \angle A + \angle C = 2$ rt. $\angle$s. (Why?)

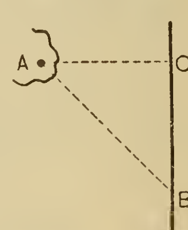
4. The angle made by the bisectors of the base angles of an isosceles triangle is equal to the exterior angle at the base.



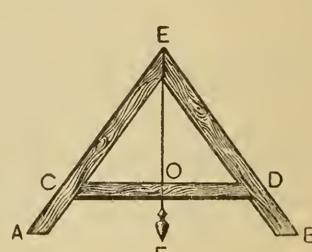
To Prove that $\alpha = \angle D$.

Suggestion. — Prove that both α and $\angle D$ are supplements of $\angle ABC$.

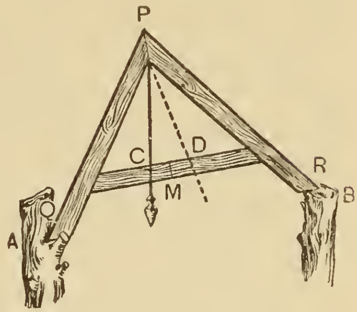
5. The distance AC at which a ship passes a lighthouse A, when moving in the direction BC, is obtained by observing the moment when the direction of the lighthouse makes an angle of 45° with the course of the ship, and again when it makes an angle of 90°, the distance the ship has gone between the two observations being noted. Show how to compute the distance at which the lighthouse is passed.



6.* An instrument for leveling called a "plumb level," used before the spirit level was invented, and used in some parts of Europe now, consists of three bars fastened together in the form of the letter A (AE, BE, CD). $AE = BE$, and $CE = DE$. A point O is marked on the tie CD, midway between C and D. A plumb line EF is attached at E. The instrument is held in an upright position, with the arms AE and BE resting upon the points to be leveled. Show that when the plumb line coincides with the point O the points A and B are on a level.

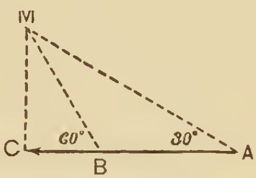


7.* To adjust a builder's plumb level that is out of adjustment: Place it upon any two supports A and B, not necessarily on a level, R on A and Q on B. Mark the point D where the plumb line cuts the crossbar. Then reverse it, placing R on B and Q on A, and mark the new point C where the plumb line cuts the crossbar. Mark the point M midway between C and D. Whenever the plumb line cuts point M, Q and R are on a level. Prove it.



SUGGESTION. — Prove $PM \perp CD$.

8. A man traveling directly west toward a city observes that a mountain peak which he knows to be directly north of the city is 30° to his right. After going 12 mi. he observes that the same peak is 60° to his right. How far is the city from his second point of observation?



9. The perpendiculars to the equal sides of an isosceles triangle from the ends of the base are equal.

10. The altitudes of an equilateral triangle are all equal.

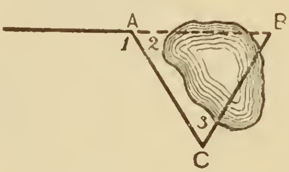
11. In an isosceles triangle the medians to the equal sides are equal.

12. The medians of an equilateral triangle are all equal.

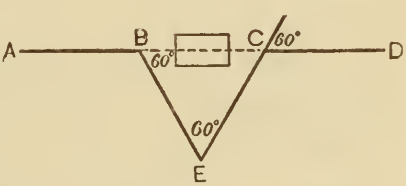
13. The bisectors of the base angles of an isosceles triangle are equal.

14. The bisectors of the three angles of an equilateral triangle are equal.

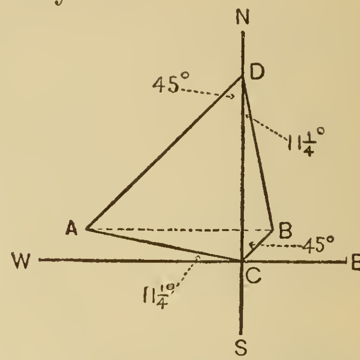
15. In surveying, the distance from A to the inaccessible point B may be obtained as follows: Run off AC until a point is found from which $\angle 3 = \frac{1}{2} \angle 1$. Measure AC. Then $AB = AC$. Prove it.



16. Justify the following method of surveying a line AB beyond an obstacle such as a building: Measure off an angle of 60° at B, and run BE sufficiently long to clear the obstacle. At E construct an angle of 60°, and measure off $EC = BE$. Then at C turn off an angle of 60°, and establish the line CD. CD is a prolongation of AB.



17.* From a point C, A is $11\frac{1}{4}°$ north of west and B is directly N.E. From D, 10 miles directly north of C, A is directly S.W. and B is $11\frac{1}{4}°$ east of south. Show that B is 10 miles directly east of A.



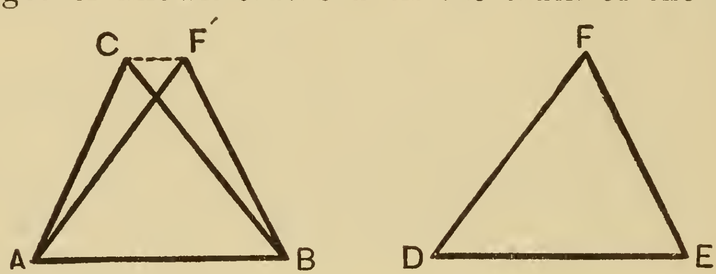
Suggestion. — In $\triangle ACB$, $\angle BCA$ is known, and $\angle CAB$ can be shown to be $11\frac{1}{4}°$. Then prove $\triangle ACB$ and CBD congruent.

Caution. — Do not assume $AB \parallel WE$ to get $\angle CAB$.

18. If one acute angle of a right triangle is double the other, the hypotenuse is double the shorter leg.

Suggestion. — Extend the shorter leg its own length through the vertex of the right angle, and connect its extremity with the other extremity of the hypotenuse. Prove the triangle thus formed isosceles.

85. Indirect method of proof. — The theorems thus far have been proved either by the **method of superposition**, or by putting together known truths until the truth of the theorem



being considered was established. This latter method is known as the **synthetic method**, or **direct method**. Another method may often be used to advantage, particularly in converse theorems. In this method we assume the conclusion of the theorem to be *not true* and prove that the assumption leads to an *absurdity*. This method is known as the **indirect method** or **reductio ad absurdum**. As an example of this method let us prove the theorem of § 81 in this way.

Proof. 1. Place $\triangle DEF$ upon $\triangle ABC$ so that DE coincides with its equal, AB.

2. Now suppose that F does *not* fall upon C, but at F'.

3. Draw CF'.

4. Now $AF' = AC$, and $BF' = BC$.

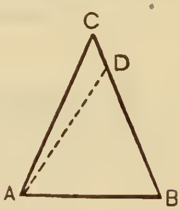
5. $\therefore$ △ ACF' and BCF' are isosceles.

6. $\therefore$ the perpendicular bisector of CF' must pass through both A and B, and must coincide with AB, which is absurd.

7. $\therefore$ since F cannot fall at any point F' not C, it must fall on C, and the two triangles are congruent.

EXERCISE

Prove the theorem of § 83 by the indirect method, or *reductio ad absurdum.*



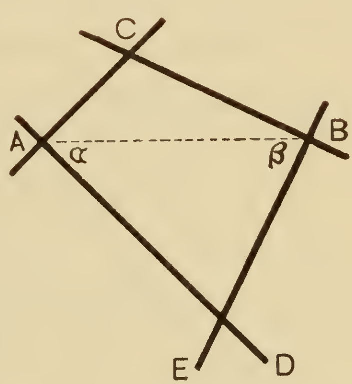
Suggestions. — 1. Assume that BC and AC are not equal, and that $BC > AC$.

2. Lay off $BD = AC$, and draw AD.

3. $\because AB \equiv AB$, $BD = AC$, and $\angle BAC = \angle CBA$, compare $\triangle ABC$ with $\triangle ABD$.

4. What conclusion follows?

86. Theorem. — *Two lines perpendicular respectively to two intersecting lines must meet.*



Hypothesis. AC and BC intersect at C; $AD \perp AC$ and $BE \perp BC$.

Conclusion. AD and BE meet.

Suggestions. 1. Assume that AD and BE do not meet.

2. Then $AD \parallel BE$.

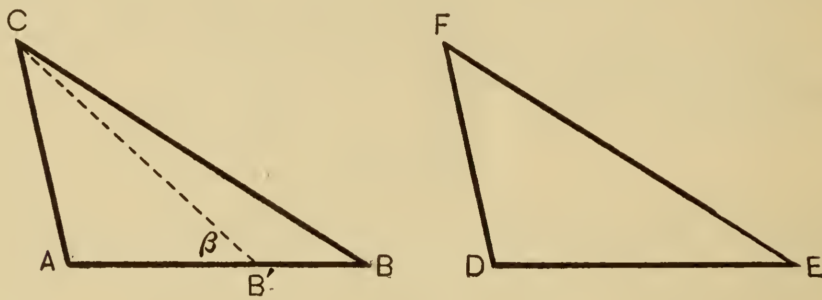
3. Now draw AB. Since it is assumed that $AD \parallel BE$, what is the sum of α and β?

4. What is the sum of $\angle DAC$ and $\angle EBC$?

5. Show that an absurdity results.

87. The ambiguous case in triangles. — We shall now proceed to the study of two triangles, *when two sides and an angle opposite one of them in one are equal respectively to two sides and the corresponding angle in the other.*

Case I. *Suppose the given equal angles not acute.*



Hypothesis. $AC = DF$, $CB = FE$, $\angle A = \angle D$, and these angles not acute.

1. Place $\triangle DEF$ upon $\triangle ABC$ so that DF will coincide with its equal AC, F falling at C and D at A.

2. Now $\angle D = \angle A$.

3. $\therefore DE$ will fall along AB.

4. If E does not fall on B, suppose that it falls at B'.

5. Then $CB' = CB$, and $\therefore \angle B = \angle CB'B$. (?)

6. But $\angle CB'B$ is obtuse, since β must be acute. (§ 63)

7. $\therefore \angle B \neq \angle CB'B$. (§ 63)

8. $\therefore$ the supposition that E does not fall on B is false, and hence the $\triangle$ coincide throughout.

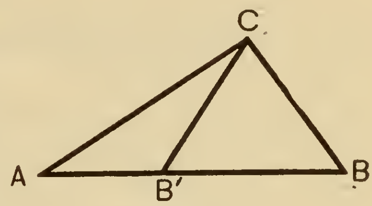
9. $\therefore \triangle ABC \cong \triangle DEF$.

Corollary. — *Two right triangles are congruent, if a leg and the hypotenuse of one equal a leg and the hypotenuse of the other.* (For the equal angles opposite the equal sides are not acute.)

Case II. *Suppose the given equal angles acute.*

Superpose, as in Case I.

In the $\triangle ABC$ and $AB'C$, in the figure, $AC \equiv AC$, $CB = CB'$, and $\angle A \equiv \angle A$, although the $\triangle$ are not congruent. $\therefore$ it is evident that if the given equal angles are acute, the $\triangle$ when applied one to the other may or may not coincide. This is known as the **ambiguous case**. It may be shown that when the sides opposite the given equal acute angles are *not less than* the other given equal sides, the case is not ambiguous. Draw figures to illustrate this case.



88. A summary of theorems on congruent triangles. — It is evident that we may now state all of the theorems on congruent triangles in one, as follows:

Two triangles are congruent when any three independent parts of one are equal to the corresponding parts of the other, except in the ambiguous case.

Note. — There are evidently only the following three cases:

I. *Two sides and an angle.*

II. *Two angles and a side.*

III. *Three sides.*

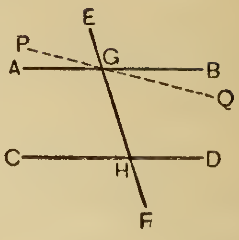
The three angles are not *independent* parts, for *if two angles of a triangle are known, all are known.* (Why?)

EXERCISES

1. Prove by the indirect method the fact formerly assumed that from a given external point only one perpendicular can be drawn to a given line.

Suggestion. — Suppose that a second perpendicular from the point could be drawn. Show that this would lead to a contradiction of the theorem that the sum of the angles of a triangle is a straight angle.

2. Prove the theorem in § 39 by the indirect method.



SUGGESTIONS.—1. Suppose AB not parallel to CD, and that PQ is drawn through G parallel to CD.

2. Then compare $\angle PGH$ and $\angle DHG$.

3. Show that the supposition leads to the conclusion that two angles are equal when one is only a part of the other.

3. Prove the theorem in § 40 by the indirect method.

4. Prove the theorem in § 41 by the indirect method.

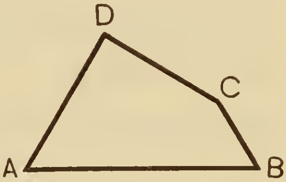
5. Prove the theorem in § 42 by the indirect method.

6. Prove the theorem in § 43 by the indirect method.

CHAPTER IV

QUADRILATERALS, POLYGONS, LOCI

89. Quadrilaterals. — A figure like the one here represented is called a **quadrilateral** because it is made of four lines. The four lines are called the **sides**, the angles which they form at A, B, C, and D, the **angles**, and the points A, B, C, and D, the **vertices**, of the quadrilateral. When no two sides are parallel, as in the figure, it is a **trapezium**. This term, however, is seldom used.

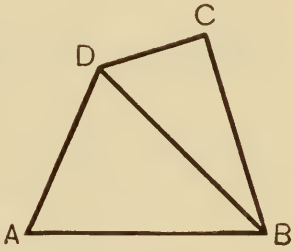


The sum of the sides of a quadrilateral, or of any other figure, is called its **perimeter**.

How many angles has a quadrilateral? How many vertices? Read the sides of this quadrilateral. The angles. The vertices.

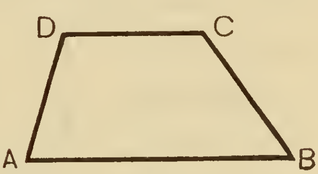
A quadrilateral is named by reading the vertices, going around the figure in a continuous direction. Name this one.

90. Diagonals. — The line BD, which joins two opposite vertices, is called a **diagonal** of $ABCD$.



It divides the figure into how many triangles? Can you draw any other diagonal? What? Draw a quadrilateral and name its diagonals.

91. Trapezoids. — A quadrilateral having one and but one pair of opposite sides parallel is a **trapezoid**.



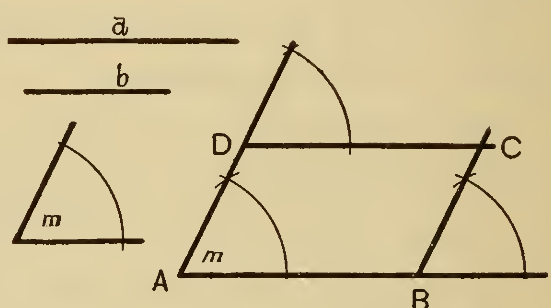
If the non-parallel sides are equal, the trapezoid is **isosceles**.

92. Parallelograms. — A quadrilateral having both pairs of opposite sides parallel is called a **parallelogram**. If the angles of the parallelogram are all *right angles*, it is called

PARALLELOGRAM RECTANGLE SQUARE RHOMBUS

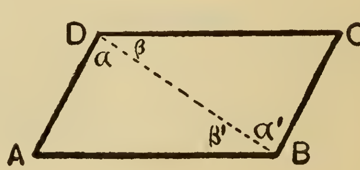
a **rectangle**; and if the sides of a rectangle are all equal, it is called a **square**. If the *sides* of a parallelogram are all equal and its angles *not* right angles, it is called a **rhombus**.

93. Construction. — A parallelogram which has two sides and the included angle equal, respectively, to two given sects a and b, and the given angle m, may be constructed as follows:



Draw an angle BAD equal to angle m. On one side mark off AB equal to a, and on the other mark off AD equal to b. Then, at D draw a parallel to AB. And at B, draw a parallel to AD. These lines meet at some point C. Then $ABCD$ is evidently the parallelogram desired.

94. Theorem. — *If two opposite sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*



Hypothesis. In $ABCD$, $AD = BC$ and $AD \parallel BC$.

Conclusion. $ABCD$ is a $\square$, *i.e.* $AB \parallel DC$.

Suggestion. Draw DB. AB may be proved parallel to DC by proving $\beta = \beta'$. How are angles proved equal?

Write out complete proof.

Could you have proved the theorem as well by drawing the diagonal AC?

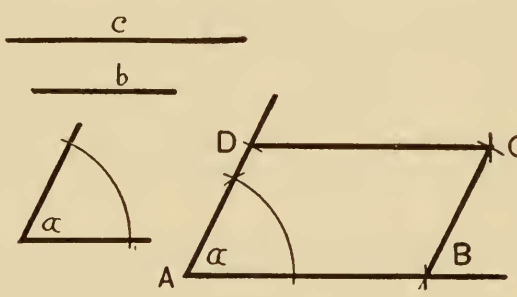
95. Theorem. — *In any parallelogram,* (1) *the opposite sides are equal;* and (2) *the opposite angles are equal.*

(Prove by means of congruent triangles.)

96. Theorem. — (Converse of § 95.) *In any quadrilateral, if* (1) *the opposite sides are equal,* or (2) *the opposite angles are equal, the figure is a parallelogram.*

Write out complete proof. Use congruent triangles for (1). Use theorems of §§ 59 and 42 for (2).

97. Construction. — The theorem of § 96 suggests a shorter method for constructing a parallelogram than that of § 93, *when we have given two sides and the included angle.*



Given. The sides b and c and the included $\angle a$ of a ▱.

Required. To construct the ▱.

Suggestions. 1. After an angle equal to the given angle has been constructed and D and B found on the sides so that $AD = b$ and $AB = c$, what remains to be found?

2. How far is C from B?

3. How far is C from D?

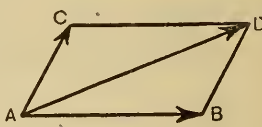
4. Where then is C?

Now give the steps of the construction, and prove that the figure is the required parallelogram.

EXERCISES

1.* Construct a parallelogram which shall have two sides and the included angle equal, respectively, to two given sects a and b, and the given angle m.

2.* If two forces are exerted in different directions upon the same object at A, they have the same effect as a single force, called their *resultant.* If the directions and magnitudes of the two forces are represented by the sects AB and AC, the direction and magnitude of the resultant will be represented by the sect AD, diagonal of the parallelogram having AB and AC as adjacent sides.



Two forces, one of 100 lb., the other of 200 lb., are exerted upon a body at an angle of 45° with each other. Representing 100 lb. by a sect 1 in. long, draw the forces to scale, and find the resultant. Use the protractor in drawing to scale.

3.* Two forces are exerted upon a body at right angles with each other, one of 400 lb., the other of 650 lb. Construct and compute their resultant, as in Exercise 2.

4.* Two forces are exerted upon a body at an angle of 120° with each other, one of 48 lb., the other of 60 lb. Find their resultant.

5.* When a train is approaching a station at a velocity of 40 ft. per second, a mail bag is thrown at right angles from the car with a speed of 20 ft. per second. Find the actual direction and speed of the moving bag.

SUGGESTION. — Construct as in case of forces.

6.* Three forces in the same plane, of 120 lb., 100 lb., and 60 lb., respectively, are exerted simultaneously upon an object. The second makes with the first an angle of 60°, and the third with the first 90°. Find their resultant.

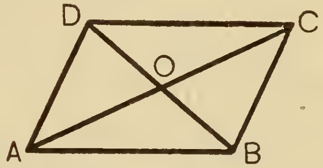
SUGGESTION. — Construct the resultant of the first two, then construct the resultant of this resultant and the third force.

7.* Four forces in the same plane, of 200 lb., 80 lb., 160 lb., and 100 lb., respectively, are exerted simultaneously upon an object. The second makes with the first an angle of 90°, the third with the first 120°, the fourth with the first 180°. Find their resultant.

98. Assumption. — It is evident that

The diagonals of any parallelogram intersect in a point which lies within the parallelogram.

99. Theorem. — *In any parallelogram, the diagonals bisect each other.*



Suggestion. Prove the theorem by proving $\triangle ABO \cong \triangle CDO$.

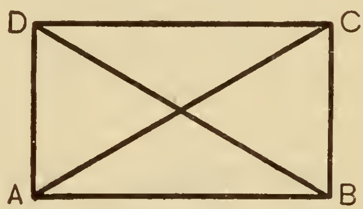
100. Theorem. — (Converse of § 99.) *If the diagonals of any quadrilateral bisect each other, the figure is a parallelogram.*

(Write out complete proof.)

101. Theorem. — *In any rectangle, the diagonals are equal.*

Suggestion. Prove $\triangle DAB \cong \triangle ABC$.

102. Theorem. — (Converse of § 101.) *If the diagonals of a parallelogram are equal, the figure is a rectangle.*

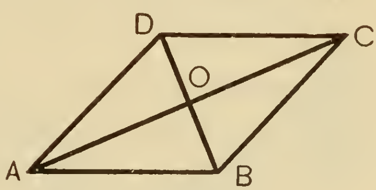


(Write out complete proof.)

103. Corollary. — *In any right triangle, the middle point of the hypotenuse is equidistant from the vertices.*

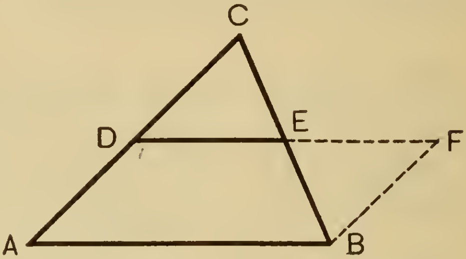
(Let $\triangle ABC$ of the above theorem be the given $\triangle$.)

104. Theorem. — *The diagonals of a rhombus bisect each other at right angles.*



Suggestion. By the definition of a rhombus, we have given $AD = DC = BC = AB$. Prove the angles at O right angles. Use congruent triangles.

105. Theorem. — *A sect drawn between the middle points of two sides of a triangle is parallel to the third side and equal to one half of it.*



Hypothesis. In $\triangle\ ABC$, DE connects the middle points of AC and BC.

Conclusion. $DE \parallel AB$ and $DE = \frac{1}{2}\ AB$.

Suggestions. 1. Since no former theorem has established a relation between one sect and one half of another, produce DE to F, making $EF = DE$, and prove $DF = AB$.

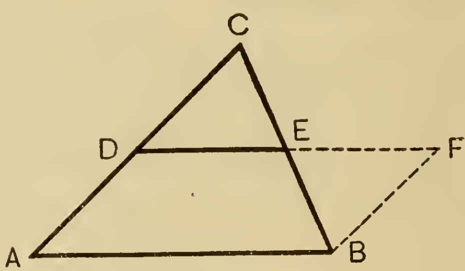
2. Compare $\triangle\ DEC$ with $\triangle\ EBF$.

3. Then compare BF with DC and hence with AD.

4. Compare $\angle\ C$ and $\angle\ FBC$, and thus the relative directions of AD and BF.

5. From § 94, what is the conclusion?

106. Theorem. — *If a straight line bisects one side of a triangle and is parallel to a second side, it bisects the third side also.*



Hypothesis. In $\triangle\ ABC$, DE bisects AC and $DE \parallel AB$.

Conclusion. DE bisects BC, *i.e.* $BE = EC$.

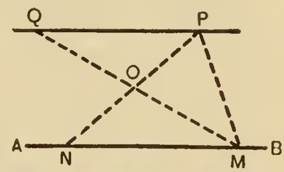
Suggestions. 1. Through B draw a parallel to AC, and prolong DE to meet it at F, forming $\square\ ABFD$.

2. Show that $\triangle\ BEF \cong \triangle\ DEC$.

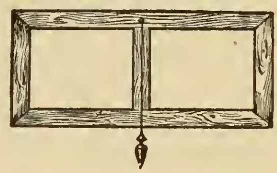
3. Hence $BE = EC$.

EXERCISES

1. Surveyors, in order to establish a second point that will make with a given point P a line parallel to a given line AB, sometimes proceed as follows: Measure off any two sects PM and PN from P to AB. Mark O the middle point of PN. Measure the distance MO, and extend it in a straight line to Q so that $OQ = MO$. Show that Q is the required point.

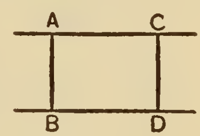


2. The figure shows an instrument sometimes used in leveling a surface. It is a rectangular frame with a plumb line hung from the center of the upper side, and the center of the lower side marked. Show that the surface is level when the plumb bob hangs directly over the center of the lower side of the frame.



3. Parallel sects comprehended between parallel lines are equal.

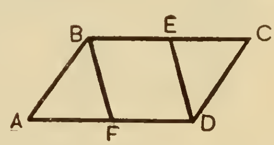
4. Two parallel lines are everywhere equally distant.



SUGGESTION. — Sects AB and CD, drawn from any two points of AC, perpendicular to BD, measure the distances of those points from BD.

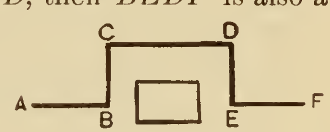
5. If one angle of a parallelogram is a right angle, all four angles are right angles and the figure is a rectangle.

6. The diagonal of a parallelogram divides it into two congruent triangles.

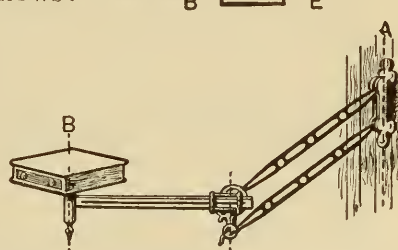


7. If $ABCD$ is a parallelogram, and E and F, respectively, the middle points of BC and AD, then $BEDF$ is also a parallelogram.

8.* In surveying, the line AB is produced beyond an obstacle such as a house as follows: At B run BC at right angles to AB; run CD at right angles to BC; run DE at right angles to CD, and equal to BC; run EF at right angles to DE. Then EF is the prolongation of AB. Prove it.

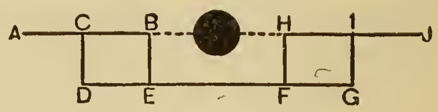


9. An adjustable bracket, as shown in the figure, is fastened to the wall at A, and carries a shelf B. Show that as the bracket is moved so that B is raised and lowered the shelf remains horizontal.



SUGGESTION. — Show that the arm which carries B is perpendicular to a line through the elbow of the bracket that remains constantly vertical, because it is parallel to a fixed vertical line, throughout the motion.

10.* In surveying, a line AB may be extended beyond an obstacle such as a tree as follows: At C and B, two points of AB, equal perpendiculars CD and BE to AB are measured off; the line DE is run sufficient to clear the obstacle; at F and G, two points of DE prolonged, FH and GI are measured equal to CD and perpendicular to DG. Then the line HJ determined by H and I is a prolongation of AB. Prove it.



11. Prove the theorem of § 105 by drawing $BF \parallel AC$ to meet DE produced in F.

12. If the diagonals of a parallelogram are perpendicular to each other, the figure is a rhombus or a square; a square, if the parallelogram is a rectangle.

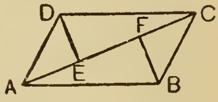
13. The diagonals of a square or rhombus are perpendicular to each other, and bisect the angles of the square or rhombus.

14. The intersection of the diagonals of a parallelogram is called the **center of the parallelogram.** Every sect through the center of a parallelogram and terminating in the sides is bisected at this point.

SUGGESTIONS. — Draw a diagonal. Prove the two parts of the given sect corresponding sides of congruent triangles.

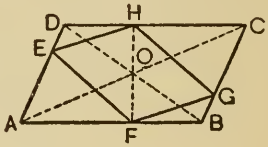
15. The perpendiculars to a diagonal of a parallelogram from the opposite vertices are equal.

SUGGESTION. — Compare $\triangle AED$ and FBC.



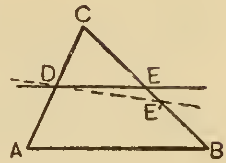
16. If the vertices of one parallelogram lie upon the four sides of another, the parallelograms have the same center.

SUGGESTIONS. — Draw the diagonals AC and BD, intersecting at O. Draw OE, OF, OG, OH, and prove EOG and FOH straight lines and hence diagonals.



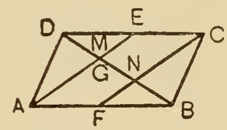
17. If from any point in the base of an isosceles triangle parallels to the sides are drawn, the parallelogram thus formed will have a constant perimeter.

Suggestion. — Prove the perimeter of the $\square$ equal to the sum of the two equal sides of the triangle.



18. Prove the theorem in § 106 by the *indirect* method.

Suggestions. — Suppose that DE does not bisect BC, and that DE' does. Then what is true of DE' by § 105? See § 44.

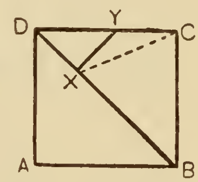


19. $ABCD$ is any parallelogram and E and F are the middle points of DC and AB, respectively. Prove that the straight lines AE and CF trisect the diagonal (cut it into three equal parts).

Suggestions. — 1. What kind of quadrilateral is $AFCE$?

2. In $\triangle ABM$, compare NB and NM. See Ex. 18.

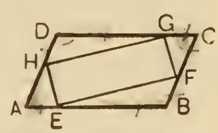
3. In $\triangle CDN$, compare DM and NM.



20. In the square $ABCD$, on the diagonal BD, BX is taken equal to BC, and XY is drawn perpendicular to BD. Prove that $DX = XY = YC$.

Suggestion. — Two sects are equal if they are sides opposite two equal angles in a triangle.

21 In the parallelogram $ABCD$, points E, F, G, and H are so taken that $AE = BF = CG = DH$. Prove that the figure $EFGH$ is a parallelogram.



Suggestions. — 1. Compare $\triangle EBF$ with $\triangle GDH$, and thus HG with EF.

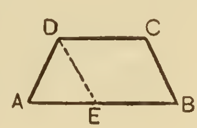
2. Compare $\triangle AEH$ with $\triangle FCG$, and thus HE with GF.

3. What conclusion follows?

22. In the same figure, if E, F, G, and H are taken so that $AE = AH = CF = CG$, prove that $EFGH$ is a parallelogram.

23. In the same figure, if $AE = CG$ and $BF = DH$, prove that $EFGH$ is a parallelogram.

24. If the non-parallel sides of a trapezoid are equal (isosceles trapezoid), the angles which these sides make with either of the parallel sides are equal.

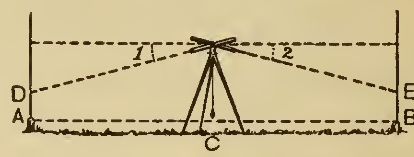


Suggestions. — Draw DE parallel to CB. Compare DE and DA. Then compare the angles at A, E, and B.

25. Conversely, if the angles made by the non-parallel sides of a trapezoid with either of the parallel sides are equal, the trapezoid is isosceles.

26. The diagonals of an isosceles trapezoid are equal.

27.* Surveyors must sometimes solve this problem: To set two stakes in the ground, on a level, by use of a transit (surveyor's instrument containing a telescope which is leveled by means of a bubble tube) whose bubble tube is not accurately adjusted. Set the instrument at C equally distant from A and B. Drive stake A, and place rod upon it. Sight the instrument at the rod, and note the reading AD on the rod. Then start the stake B, set the rod upon it, and revolve the telescope until it sights at the rod again at E. Now drive stake B down until $BE = AD$. Then A and B are on a level. Prove it.



28. Two parallelograms are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.

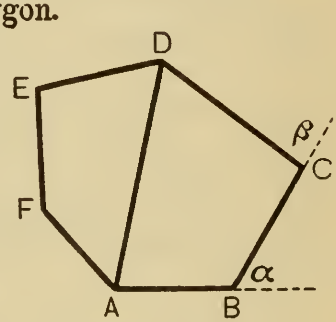
Suggestion. — Superpose one upon the other. Use the assumption in § 44.

29. Two rectangles are congruent if two adjacent sides of one are equal, respectively, to two adjacent sides of the other. (Use Ex. 28.)

POLYGONS

107. Polygons. — The figure in a plane formed by a broken line whose end points coincide is a **polygon.**

The sects which compose the broken line are the **sides** of the polygon, and the points where the sides meet are the **vertices** of the polygon. The angles formed by the adjacent sides are the **interior angles**, or merely the **angles**, of the polygon.



Convex Polygon

It is evident that a polygon has just as many vertices and as many angles as it has sides.

A polygon is named by naming its vertices, taken in order, as *ABCDEF*, in the figure.

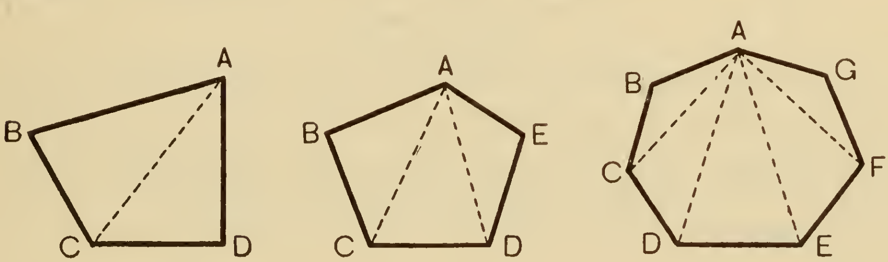
A sect joining two non-adjacent vertices, such as *AD*, is a **diagonal.**

A polygon no side of which produced will enter the surface inclosed is **convex.**

If at each vertex of a polygon one of the sides is produced, in an order the same for all vertices, the angles such as α and β thus formed are **exterior angles.**

The triangle and quadrilateral with which the student is familiar are particular polygons with three and four sides, respectively.

108. Theorem. — *The sum of the interior angles of a convex polygon of n sides is $n - 2$ straight angles.*



Hypothesis. Polygon $ABC\cdots$ has n sides.

Conclusion. The sum of the angles of $ABC\cdots$ equals $n - 2$ st. $\angle$s.

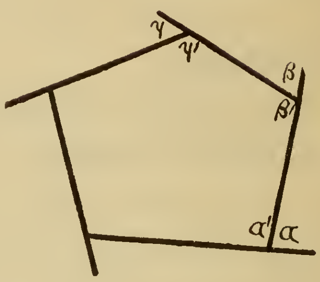
Suggestions. 1. Draw all the diagonals possible from any one vertex, as A.

2. Show that $n - 2$ triangles are thus formed.

3. What is the sum of the angles of each triangle?

4. Show that the sum of all the angles of all the triangles equals the sum of the angles of the polygon.

109. Theorem. — *The sum of the exterior angles of any convex polygon is 2 straight angles.*



Hypothesis. α, β, γ, etc., are the exterior angles of any polygon of n sides.

Conclusion. $\alpha + \beta + \gamma + \cdots = 2$ st. $\angle$s.

Proof. 1. Let α', β', γ', etc., be the interior angles adjacent to α, β, γ, etc.

2. Then $\alpha + \alpha' =$ st. $\angle$, $\beta + \beta' =$ st. $\angle$, $\gamma + \gamma' =$ st. $\angle$, etc.

3. $\therefore \alpha + \alpha' + \beta + \beta' + \gamma + \gamma' + \cdots = n$ st. $\angle$s.

4. But $\alpha' + \beta' + \gamma' + \cdots = n - 2$ st. $\angle$s.

5. $\therefore \alpha + \beta + \gamma + \cdots = 2$ st. $\angle$s. (Subtracting.)

EXERCISES

1. Prove the theorem in § 108 by drawing lines from any point O within the polygon to each of the vertices.

Suggestion. — The sum of the angles of the polygon is obtained by subtracting the sum of the angles about O from the sum of all the angles of the triangles.

2. Find the sum of the angles of a polygon of four sides; of five sides; of six sides; of seven sides; of eight sides; of ten sides; of twelve sides; of twenty sides.

3. The sum of the angles of a polygon is 14 rt. $\angle$s. How many sides has the polygon?

4. Show that if all of the angles of a polygon of n sides are equal, each angle equals $\frac{n-2}{n}$ st. $\angle$s.

5. How many degrees in each angle of an equiangular polygon of five sides?

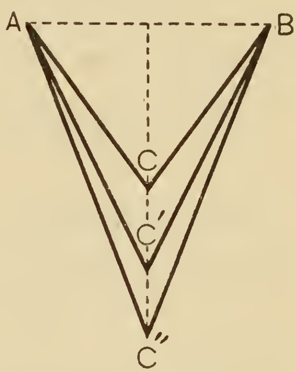
6. Each angle of an equiangular polygon is 150°. How many sides has it?

7. If two angles of a quadrilateral are supplementary, show that the other two are supplementary also.

8. How many tiles, each with six equal sides and its angles all equal, can be placed corner to corner in a floor so as completely to cover the floor? Why?

LOCI

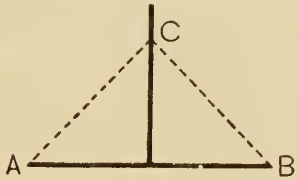
110. The loci of points. — Sometimes it is convenient to think of a figure as having been formed by a point moving under certain conditions. The figure thus formed is called the **locus** (or path) of the point. *This figure, then, must contain all the points that satisfy the given conditions, and no others.*

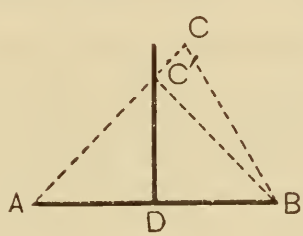


If a man walks along by a straight road, always keeping just 10 yards from the road, what is his path or *locus?*

Suppose that a man, holding two tape measures, one fastened at A and the other at B, should walk along $CC'C''$ so as to keep equidistant from A and B, what is his path or locus?

111. Theorem. — *Every point on the perpendicular bisector of a sect is equidistant from the ends of the sect.*





(Prove $AC = CB$ by congruent triangles.)

112. Theorem. — *Every point without the perpendicular bisector of a sect is unequally distant from the ends of the sect.*

(Prove that since $CC' + C'B$ is greater than CB (Why?), then AC is greater than CB.)

113. Remark. — It is evident from the definition of § 110 and from the *two* theorems of §§ 111 and 112, that

The locus of points equidistant from two points is the perpendicular bisector of the sect joining the points.

Note. — It should be observed that to prove that the locus of points is a line or sets of lines, it is sufficient to show that

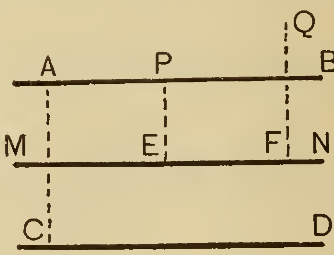
(1) *Every point in the line (or set of lines) fulfills the required condition or conditions; and that*

(2) *No other point (without the line, or set of lines) does fulfill the condition or conditions.*

In place of proving these two things, it would be sufficient to prove (1) above and

(3) *Every point that fulfills the required condition or conditions is in the line (or set of lines).*

114. Theorem. — *The locus of all points at a given distance from a straight line is a pair of straight lines on opposite sides of the given line, parallel to it, and at the given distance from it.*



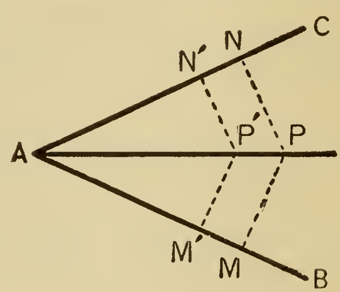
This requires the proof of what two theorems?

Suggestions. 1. Let the given distance be d, and the given line be MN.

2. Draw $AC \perp MN$, making $AM = MC = d$.

3. Then draw $AB \parallel MN$ and $DC \parallel MN$.

4. If P is any point on either AB or CD, and Q any point on neither AB nor CD, prove $PE = d$ and $QF \neq d$.



115. Theorem. — *The locus of points within an angle and equidistant from the sides is the bisector of the angle.*

What two theorems must be proved?

Suggestions. 1. Let P be any point in the bisector AP. Prove P equidistant from AB and AC.

2. Let P' be any point within the angle and equidistant from AB and AC. Prove AP' the bisector of $\angle BAC$.

EXERCISES

1. Find a point equally distant from three given points.

(Upon what two loci must it lie?)

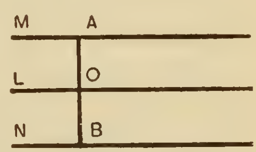
2. Find a point within a triangle equally distant from the three sides. (Use § 115.)

3. A point is to be equally distant from two given points and also equally distant from the sides of a given angle. Locate it.

(Upon what two loci must it lie?)

4. A robber buried some loot near the crossing of two straight railroad tracks, equally distant from the two tracks and 50 ft. from one of them. Show how to locate it.

5. The locus of points that are equally distant from two parallel lines is a line parallel to each of them and passing through a point midway between them.



Suggestion. — Let AB be a common perpendicular to the parallel lines M and N. (By what authority?) Let $OA = OB$. Through O draw L parallel to either M or N. Prove that any point on L is equally distant from M and N, and that any point not on L is not equally distant from M and N.

6. A point is to be equally distant from two given parallel lines and also equally distant from the sides of a given angle. Locate it.

Under what condition is the construction impossible?

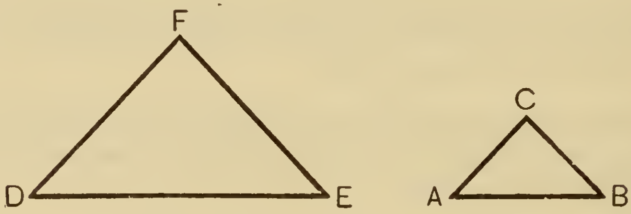
7. A point is to be equally distant from two given points and also equally distant from two given parallel lines. Locate it.

When is the construction impossible?

CHAPTER V

SIMILAR TRIANGLES

116. Similar triangles. — Draw any triangle ABC. On a straight line take a sect DE twice as long as AB. Form a triangle upon DE as base with angles equal to $\angle A$ and $\angle B$,



respectively. Since two angles of one triangle are equal by construction to the two of the other, what must be true of the third angles? Why?

Two triangles in which each angle of one is equal to the corresponding angle of the other are called **mutually equiangular**.

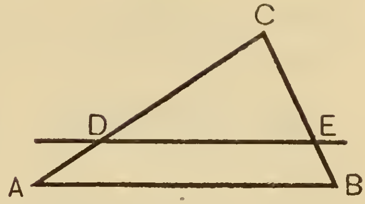
What can you say of the shapes of triangles ABC and DEF?

When two triangles have the angles of one respectively equal to the angles of the other, the two triangles have the same shape, and hence are called **similar**.

In written work, the words "is similar to" are often expressed by the symbol $\sim$.

117. Theorem. — *Any straight line parallel to a side of a triangle forms with the other two sides a triangle similar to the first.*

Hypothesis. DE is any straight line $\parallel AB$, in $\triangle ABC$, forming $\triangle DEC$.



Conclusion. $\triangle DEC \sim \triangle ABC$.

(Prove the triangles mutually equiangular.)

118. Remark. — The above theorem is true if DE intersects the sides produced.

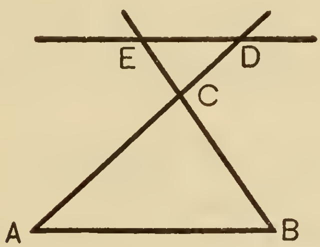


FIG. 1.

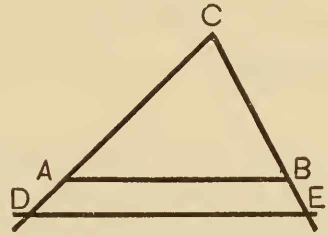
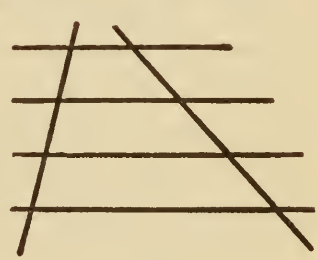


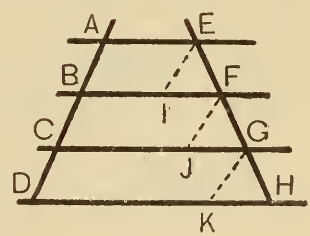
FIG. 2.

Let the student prove the theorem with Figs. 1 and 2.

119. Parallels cut by transversals. — Draw a system of parallel lines equidistant from each other. Cut them by two or more transversals. With your dividers compare the sects intercepted on each transversal by the parallels. What is your conclusion?



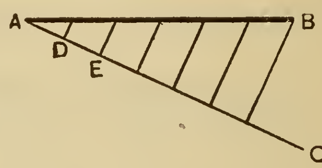
120. Theorem. — *If a system of parallel lines divides any transversal into equal sects, it divides every transversal into equal sects.*



Suggestion. Let $AB = BC = CD$. Prove $EF = FG = GH$, by parallelograms and congruent triangles.

121. Corollary. — *If a system of lines, parallel to a side of a triangle, divides one side of the triangle into equal sects, it divides the other side into equal sects also.*

122. Construction. — *Divide a given sect into any number of equal parts.*



The construction is based upon § 121.

Given the sect AB.

Required to divide AB into any number, say 6, equal parts.

Construction. 1. Draw AC at any convenient angle with AB.

2. On AC mark off 6 equal sects, AD, DE, etc.

3. Join B to the last point of division of AC.

4. Through the other points of division of AC draw lines parallel to this line drawn from B.

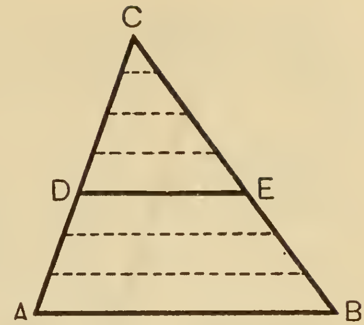
5. These parallels will divide AB into 6 equal parts. (The proof is left to the student.)

123. Numerical measures of geometrical magnitudes. — The **numerical measure** of a geometrical magnitude, such as a sect, is the number of times that it contains a given unit of measure. Thus, if a sect AB is divided into 6 equal sects, and one of these equal parts taken as the unit of measure, the numerical measure of AB is 6.

The **ratio** of two geometrical magnitudes is the quotient of their numerical measures, when the same unit of measure is applied to each.

124. Theorem. — *A line parallel to one side of a triangle divides the other two sides into sects with the same ratio.*

Hypothesis. In $\triangle ABC$, $DE \parallel AB$, and DE cuts AC in D and CB in E.



Conclusion. $\frac{AD}{DC} = \frac{BE}{EC}.$

Proof. 1. Suppose that a common measure may be found that will be contained in AD and DC m and n times, respectively.

2. Then lines through the points of division of AC, and parallel to AB, will divide BE and EC into m and n equal parts, respectively (§ 121).

3. $\therefore \frac{AD}{DC} = \frac{m}{n}$, and $\frac{BE}{EC} = \frac{m}{n}$.

4. $\therefore \frac{AD}{DC} = \frac{BE}{EC}$.

NOTE. — This proof is incomplete, for it assumes that the two terms of the ratio $\frac{AD}{DC}$ have a common measure. That is, it assumes that AD and DC are *commensurable*. This need not be the case. There are sects, such as those having the relative lengths of a side and diagonal of a square, which have no common measure. The *case of incommensurable sects* is too difficult to be treated in this book.

125. Corollary. — Since $\frac{AD}{DC} = \frac{BE}{EC}$,

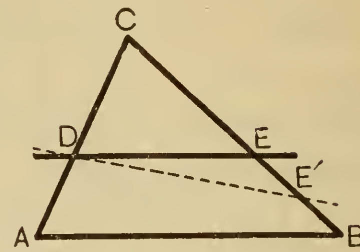
$$\frac{AD}{DC} + 1 = \frac{BE}{EC} + 1, \text{ or } \frac{AD + DC}{DC} = \frac{BE + EC}{EC}.$$

That is, $$\frac{AC}{DC} = \frac{BC}{EC}.$$

Hence, *A line parallel to a side of a triangle cuts off from the given triangle a triangle whose two sides including the common angle have the same ratio as the homologous sides of the given triangle.*

NOTE. — By **homologous sides** is meant the sides opposite equal angles.

126. Theorem. — *If a straight line divides two sides of a triangle into sects with equal ratios, it is parallel to the third side.*



Hypothesis. In the $\triangle ABC$ the straight line DE cuts the sides in D and E, so that $\frac{AD}{DC}=\frac{BE}{EC}$.

Conclusion. $DE \parallel AB$.

Proof. 1. Suppose DE not parallel to AB, but that $DE' \parallel AB$.

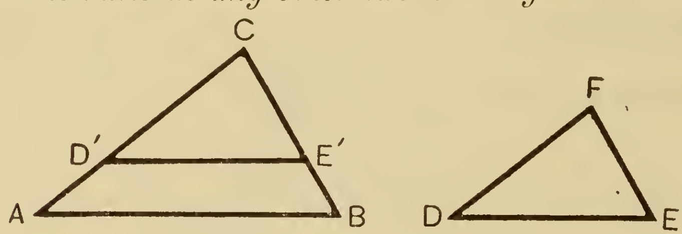
2. Then $\frac{AD}{DC}=\frac{BE'}{E'C}$, or $\frac{AC}{DC}=\frac{BC}{E'C}$. (§ 125)

3. But $\frac{AD}{DC}=\frac{BE}{EC}$, or $\frac{AC}{DC}=\frac{BC}{EC}$.

4. $\therefore \frac{BC}{E'C}=\frac{BC}{EC}$, and $E'C=EC$, which is impossible.

5. $\therefore$ the supposition is false, and hence $DE \parallel AB$.

127. Theorem. — *Two homologous sides of similar triangles have the same ratio as any other two homologous sides.*



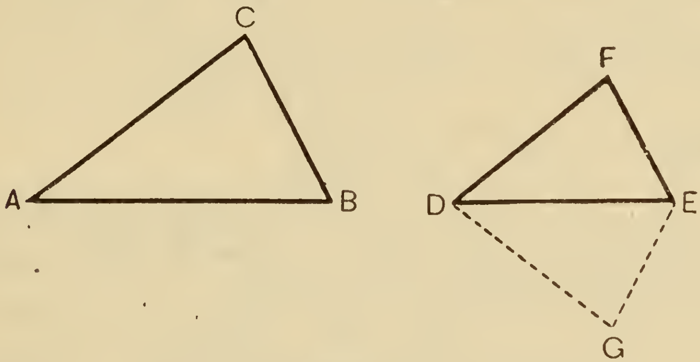
Suggestion. Suppose $\triangle DEF$ placed upon $\triangle ABC$, becoming $\triangle D'E'C$, as shown in the figure, then use § 124. In how many ways may $\triangle DEF$ be placed upon $\triangle ABC$?

128. A proportion. — When two ratios are equal, they are said to form a **proportion**. Thus, in the preceding theorem, $\frac{AC}{DF}=\frac{BC}{EF}$ is a proportion. The magnitudes concerned in

the ratios are said to be **in proportion** or **proportional.** Thus, the theorem of § 127 may be stated as follows: *In two similar triangles, the homologous sides are proportional.*

129. Theorem. — *If two triangles have their corresponding sides proportional, the triangles are similar.*

Hypothesis. $\frac{AC}{AB}=\frac{DF}{DE};\ \frac{AB}{BC}=\frac{DE}{EF};$ etc.



Conclusion. $\triangle ABC \sim \triangle DEF.$

Proof. 1. At D construct $\angle GDE = \angle A$, and at E construct $\angle DEG = \angle B$. Then $\angle G = \angle C$. (Why?)

2. $\triangle ABC \sim \triangle DGE.$ (?)

3. $\therefore \frac{AC}{AB}=\frac{DG}{DE}.$ (?)

4. But $\frac{AC}{AB}=\frac{DF}{DE}.$

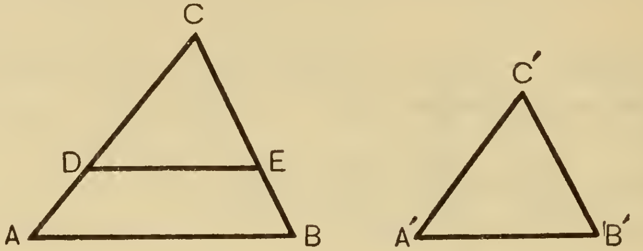
5. $\therefore \frac{DG}{DE}=\frac{DF}{DE}$, and hence $DG = DF.$

6. Similarly, $GE = EF.$

7. $\therefore \triangle DGE \cong \triangle DEF.$

8. Hence, $\triangle ABC \sim \triangle DEF.$

130. Theorem. — *If two triangles have an angle of one equal to an angle of the other and the including sides proportional, they are similar.*



Hypothesis. $\angle C = \angle C'$ and $\dfrac{CA}{C'A'} = \dfrac{CB}{C'B'}$.

Conclusion. $\triangle A'B'C' \sim \triangle ABC$.

Proof. 1. $\because \angle C = \angle C'$, $\triangle A'B'C'$ can be placed upon $\triangle ABC$ so that $A'C'$ will fall on AC and $C'B'$ on CB.

2. Let A' fall at D and B' fall at E.

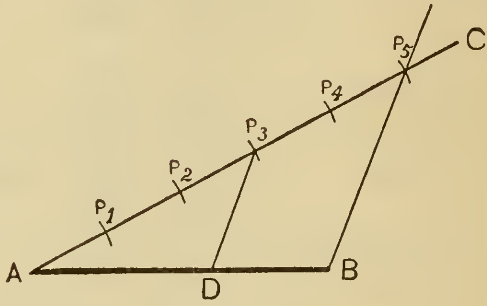
3. $DE \parallel AB$. (Why?)

4. $\therefore \triangle DEC \sim \triangle ABC$.

5. But $\triangle DEC \cong \triangle A'B'C'$. (Why?)

6. $\therefore \triangle A'B'C' \sim \triangle ABC$.

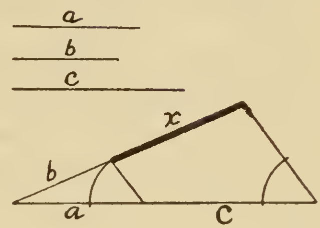
131. To divide a sect into sects of a given ratio. — The theorem of § 124 suggests a method of dividing a sect into sects of any given ratio. The method is as follows:



Suppose the given sect AB is to be divided into sects whose ratio is $3:2$. On any ray through A, as AC, lay off from A, using any convenient unit, five equal sects. Connect the last point of division with B. Now through P_3 draw a line parallel to BP_5, intersect-

ing AB. Let the point of intersection be D. Then $\frac{AD}{DB}=\frac{3}{2}$. (Why?)

132. To find a fourth proportional. — In the proportion $\frac{a}{b}=\frac{c}{x}$, x is the **fourth proportional** to a, b, and c. The fourth proportional to three given lines may be found by means of § 124. Study the construction and show how it is made. Prove that the solution is correct.



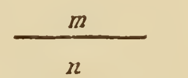
EXERCISES

1. Divide a given sect into three equal parts; into five equal parts.

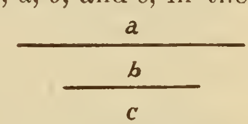
2. Divide a sect into three parts whose ratios are 1 : 2 : 3.

3. Divide a sect into two parts, one of which is $1\frac{1}{2}$ times the other.

4. Take a sect not exceeding 3 in. long and divide it into two sects whose ratio is the same as that of m and n in the margin.

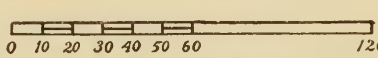


5. Find the fourth proportional to the three sects, a, b, and c, in the margin.



6. Find the fourth proportional to a, $2a$, and $3a$, when a is the sect given in Exercise 5.

7.* Drawings are not usually made of the same size as the object which they represent, but are made to *a scale*. Thus, a map of New England may be drawn to a scale of 60 mi. to the inch. A drawing representing the scale usually accompanies the map.



SCALE OF MILES.

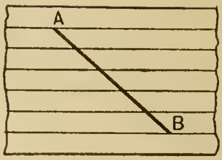
Construct a scale of 60 mi. to the inch as follows: Draw two parallel sects about $\frac{1}{12}$ in. apart, and 2 in. long. Mark off the 2 in., then divide one of them into six equal parts. Rule and mark as in the figure.

8.* Construct a scale of 100 mi. to the inch, dividing one of the half inches into five equal parts. Rule and mark as in the preceding problem.

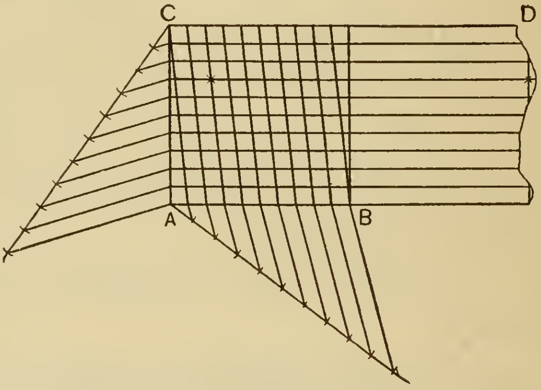
9.* Explain how the accompanying design is constructed. Make a copy of this design.

10. Show that to divide sect AB into any number of equal parts, we may draw it diagonally across a piece of squared or ruled paper as in the figure. Thus, in the figure the sect AB is cut into five equal parts.

11. Draw a sect on thin paper (tracing paper), then by adjusting it over a sheet of your notebook, and marking the points of intersection with the parallel lines of the notebook, divide it into five equal parts. Divide another sect into seven equal parts.



12. Construct a diagonal scale that will measure hundredths of an inch.



Suggestions. — Mark off a rule 1 in. wide into inches, making $AB = 1$ in. Mark off AB into ten equal parts, and AC into ten equal parts. How? Then through the points of division, draw the two sets of parallels as shown in the figure. Use your triangle in ruling the parallels.

What is the distance between the two points marked x--------x?

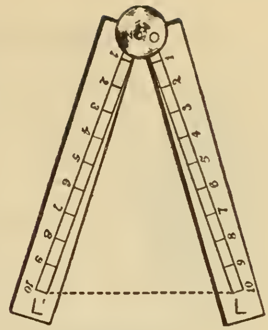
This scale should be constructed on good cardboard. *Preserve it to be used in making accurate measurements in future problems.*

13. Show how by use of the diagonal scale to measure .3 in.; .56 in.; .75 in.; 1.4 in.; 3.81 in.; .35 in.; .82 in.; 2.67 in.

14. Draw a triangle and measure the lengths of its sides to hundredths of an inch by use of the diagonal scale. In measuring, adjust the dividers to the side of the triangle, then apply them to the diagonal scale. Measure other things about you until proficient in the use of the diagonal scale.

15. The instrument represented in the figure is called a *sector*. By means of it various constructions can be made. Thus, to bisect a sect

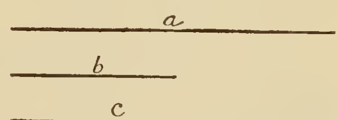
open the sector until the transverse distance from 10 to 10 on the scales OL and OL' equals the given sect. Then the distance from 5 to 5 on these scales is equal to one half of the sect. Why?



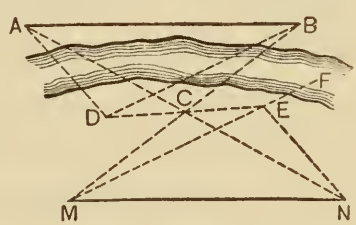
16. The sector may be used to divide a sect into any number of, say 6, equal parts as follows: Open the sector until the distance from 6 to 6 on the scales equals the given sect. Then the transverse distance from 1 to 1 will be one sixth of the given sect. Why?

17. Show how by use of the sector to divide a given sect into 9 equal parts.

18. The sector may be used to find the fourth proportional to three given sects, a, b, and c, as follows: From center O on OL, mark off $OA = a$. Open the sector until the transverse distance at A equals b. Then if OB be marked off equal to c, the transverse distance at B is the required fourth proportional. Why?



19. In surveying, it is required to run a line through a given point M parallel to a given inaccessible line AB. Show that it may be done as follows: Locate any point C in line MB. Locate point D at any convenient place. Run line $MF \parallel DB$. Find E in MF, in line with D and C. Run $EN \parallel AD$, meeting AC produced at N. Then $MN \parallel AB$. Why?



20. Prove that *in two similar triangles homologous altitudes are proportional to any two homologous sides.*

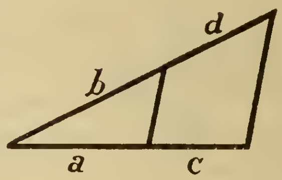
21. Draw a square with a side of 1 inch. Draw its diagonal. By use of dividers and diagonal scale, measure the length of diagonal to hundredths of an inch. What, then, is the ratio of the diagonal of a square to its side?

22. Draw a sect 1.23 in. long. One 2.07 in. long. One 0.89 in. long.

23. Construct an equilateral triangle whose side is 2.37 in.

24. Construct a triangle whose sides are 2.62 in., 2.74 in., and 1.95 in.

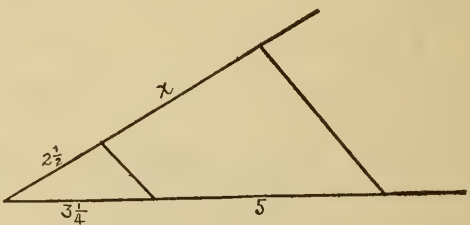
133.* **Products and quotients represented by lines.** — The preceding sections suggest an interesting method of representing a product or quotient by means of a line, *i.e.* **graphically.** Since $\frac{a}{b} = \frac{c}{d}$, when any three terms are known, the other may be found. For example, $a = \frac{bc}{d}$. Find a when $b = 3$, $c = 4$, and $d = 6$.



EXAMPLE. — To find $\frac{2\frac{1}{2} \times 5}{3\frac{1}{4}}$ we may lay off on two rays the segments as shown in the figure.

Evidently $\frac{x}{2\frac{1}{2}} = \frac{5}{3\frac{1}{4}}$, or $x = \frac{2\frac{1}{2} \times 5}{3\frac{1}{4}}$.

Hence, the length of x is the value required. This may be measured by use of the diagonal scale.



EXERCISES *

1. Find in this way $\frac{2 \times 6}{4}$
2. Find $3\frac{1}{2} \times 6\frac{3}{4}$. (Here the divisor must be considered 1.)
3. Find $6\frac{3}{4} \div 2\frac{1}{2}$. (Here $6\frac{3}{4}$ is the same as $6\frac{3}{4} \times 1$.)
4. Divide $4\frac{3}{4}$ by $1\frac{1}{2}$. $8\frac{3}{4}$ by $3\frac{1}{2}$. $6\frac{7}{8}$ by $3\frac{3}{4}$.

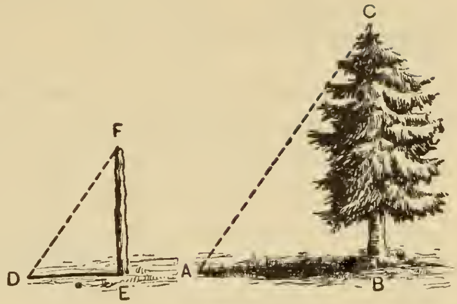
NOTE. — Use the diagonal scale in measuring the following.

5. Find 6.4×5.2.
6. Multiply 3.24 by 4.8.
7. Divide 5.75 by 2.69.

134. Finding heights and distances. — The proportion between the sides of similar triangles, established in § 127, gives a useful means of finding heights and inaccessible distances in practical work. Some of the applications will be found in the following problems.

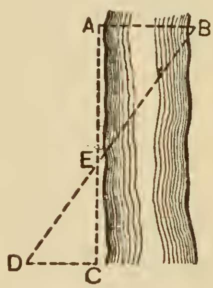
EXERCISES

1. Thales, a Greek philosopher and mathematician, about 600 B.C., is said to have amazed the Egyptians by measuring the heights of the pyramids by the length of the shadows which they cast. Measure the height of a tree as follows: —



Hold a stick, whose length is known, in a vertical position, and mark the end of its shadow. Measure the length of the shadow of the stick and also of the shadow of the tree. From these measurements, compute the height of the tree.

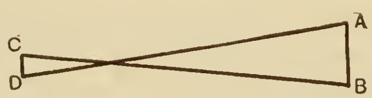
2. The distance across a stream may be estimated as follows: Find two points A and B, directly opposite each other on the two banks of the stream. Measure off a line AC, several yards long, along the bank (at right angles to AB). At C, measure off a distance CD at right angles to AC. By sighting from D to B, locate a stake at a point E of AC in line with D and B. Get the lengths of AE and EC. Triangles ABE and DEC are similar right triangles. Why? From the measurements of AE, EC, and CD, the length of AB may be computed.



Suppose that AE is 100 yd., EC 80 yd., and CD 40 yd. How wide is the stream?

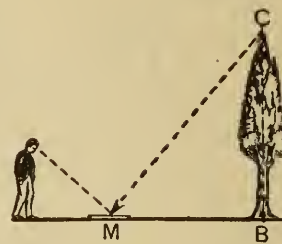
With a tapeline, measure some distance in the neighborhood in this way.

3. Sailors and others use the following method of estimating the distance DA to an object A. With the left eye closed, the finger is pointed, at arm's length, towards A. Then the right eye is closed and the left eye opened, when the object appears to have moved through the distance AB. The distance AB, being transverse to the line of sight, is estimated. The distance CD between the eyes is about one tenth of the distance from the eye to the end of the outstretched finger. If AB is 500 ft., what is the distance to A? If AB is apparently 12 ft.? Estimate in this way the distances to objects about you.



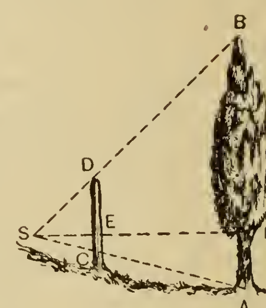
4. Find the distance between the pupils of your eyes, and the distance from your eyes to the end of your forefinger, and use the method of Exercise 3 to find distances.

5. A crude method of measuring the height of an object, in case more accurate instruments are not at hand, is by the use of a mirror. To find the height of an object BC, I place a mirror horizontally on the ground at M, and stand at the point at which the image of the top of the object is visible in the mirror. Show how, by measuring certain distances, I can compute the height of the object.



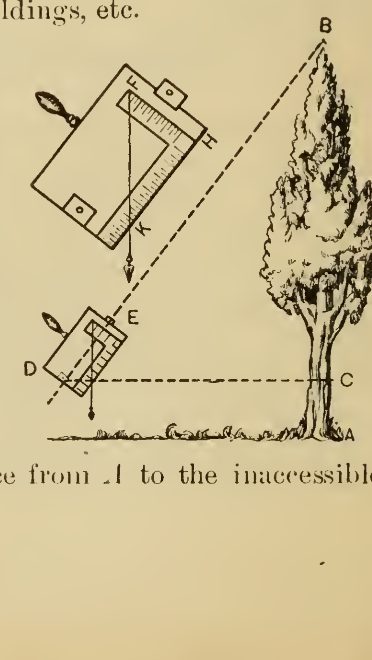
NOTE. — Light is reflected from the surface of a mirror at an angle equal to the angle at which it strikes it.

6. In forestry, when shadows cannot be used, the height AB of a tree is found as follows: A staff is held in an upright position CD. A man at S sights across the staff to the foot and to the top of the tree. An assistant notes the points C and D where the lines of vision cross the staff, and measures CD. If $CD = 4$ ft., $SE = 3$ ft., and $SF = 49$ ft. 6 in., find AB. If $CD = 4$ ft. 2 in., $SE = 2$ ft. 8 in., and $SF = 37$ ft. 5 in., find AB.

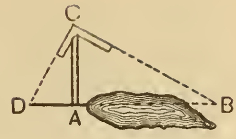


7. Use the method of Exercise 6 to find heights of things in the neighborhood, as trees, telephone poles, buildings, etc.

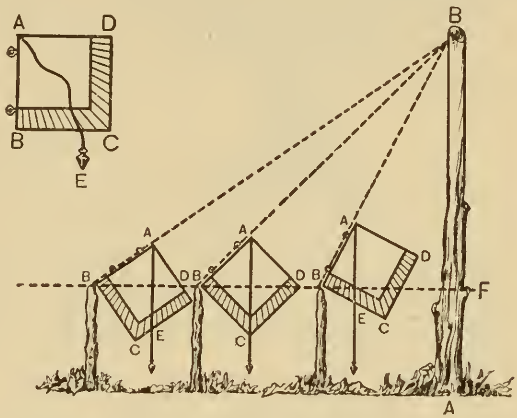
8. In Faustman's "height measure" used by foresters in finding the heights of trees, the edge of the frame HK and the sliding rule FH are marked off to a scale. FK is a plumb line. In the similar triangles FHK and DCB, show that FH corresponds to DC. When DC is 50 ft., and FH set at 5, HK is 7. Find BC. When, with another tree, $DC = 64$ ft., and $FH = 6$, $HK = 10$. Find BC. When $DC = 42$ ft., and $FH = 3\frac{1}{2}$, $HK = 6$. Find BC.



9. In the sixteenth century the distance from A to the inaccessible



point B was determined by erecting a vertical staff AC, and placing upon this an instrument resembling a carpenter's square, directing one arm towards B, and noting on the ground the point D toward which the other arm pointed. By measuring AC and DA, and using similar triangles, AB was estimated. If $AC = 5$ ft. and $DA = 6$ in., what is AB? If $AC = 6$ ft. and $DA = 4$ in., what is AB?

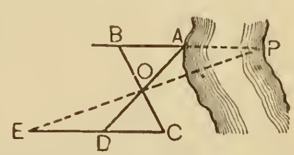


10.* A very simple way to find a height is by use of a square $ABCD$, with a plumb line AE suspended from A, and BC and DC divided into 10, 100, or 1000 equal parts.

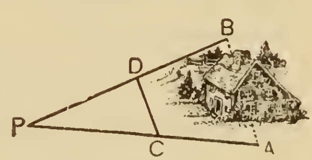
Study the diagram and show how to use the square in each case.

11.* Make an instrument like the one described in Exercise 10 and use it in finding heights of objects in the neighborhood.

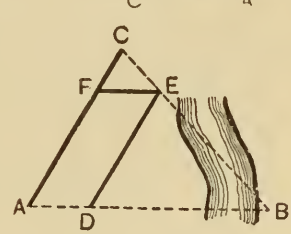
12.* To find the distance across a river from A to P, a surveyor took AB in line with AP. From A a line through O, until $OA = OD$, was taken. Then from B a line through O, so that $OC = OB$, was taken. Now on DC extended through D a point E in line with OP was taken. Show that $ED = AP$.



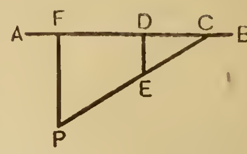
13. The distance between two points A and B separated by an obstacle may be measured as follows: From a convenient point P measure the distances PA and PB. Then in these lines locate C and D, respectively, so that $\frac{PC}{PD} = \frac{PA}{BP}$. Then measure CD. Show how AB may now be computed.



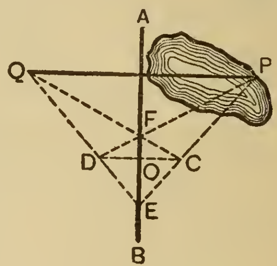
14.* To determine the distance from A to the inaccessible point B, align A, D, and B. Run DE at random. Run $AC \parallel DE$. Align C, E, and B. Measure off $FA = ED$. Measure CF, EF, CA. Show how to compute AB, and justify the method.



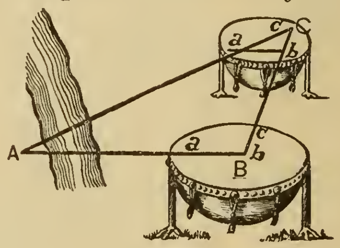
15. To lay out from a point P in a field a line at right angles to a distant line AB: Measure off any trial line PC to AB. Measure a convenient distance CD along AB. At D run a line DE at right angles to AB until E falls in line with P and C. Measure EC. Compute the fourth proportional to EC, DC, and PC. Measure off CF on AB equal to this fourth proportional. Run the line from P to F. Prove that PF is at right angles to AB.



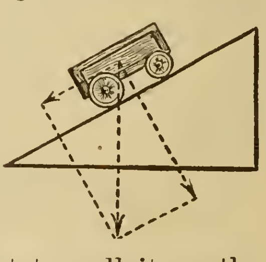
16.* To locate the perpendicular from an inaccessible point P to a given line AB: At any point O in AB, lay off $CD \perp AB$, making $OC = OD$. In AB locate E in line with P and C, and F in line with P and D. Then establish the point Q in line with C and F, and with D and E. Then PQ is the required perpendicular. Why?



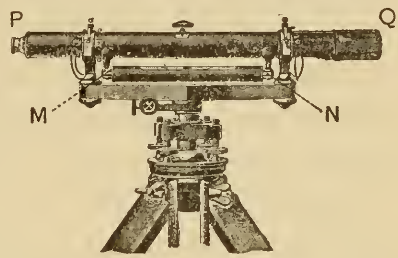
17. In the sixteenth century, an inaccessible distance AB was measured by use of drumheads as follows: On a drumhead placed at B a ray ba was drawn towards A, and another towards an accessible point C. BC was then measured, and the drumhead removed to C and placed with bc in the direction BC, such that bc represented to some scale the distance BC. Then a third ray ca was drawn towards A. Show how from such measurements it was possible to compute AB. If $BC = 200$ yd., $bc = 10$ in., and $ba = 16$ in., compute AB.

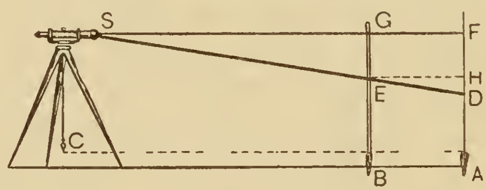


18.* When a wagon stands upon an incline, its weight is resolved into two forces, one the pressure against the incline, the other tending to make it run down the incline. Show that the force along the incline is to the weight of the wagon as the height of the incline is to its length. If the incline makes 30° with horizontal, with what force does a loaded wagon weighing three tons tend to run down the incline, *i.e.* ignoring friction, what force must a team exert to pull it up the slope?



19.* To adjust the long bubble tube MN on an engineer's transit (telescope mounted on a tripod and used for leveling and other work): The problem is to adjust the telescope PQ horizontally, then while in that position adjust the bubble tube MN by means of set-screws so that the bubble stands in the middle of the tube. To adjust the telescope horizontally, proceed thus: Place two pegs, A and B, on a level, 100 ft. or more apart. Set the transit at C in line with A and B, without leveling it. Let the line of sight cut the rod at E, when the rod is placed at B, and at D, when placed at A. Suppose SF the true horizontal. To determine the position of F, draw $EH \parallel BA$.



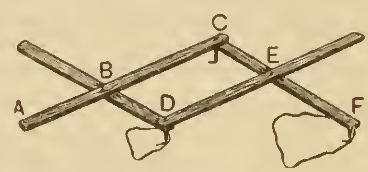


Show that $\dfrac{AB}{AC} = \dfrac{DH}{DF}$.

Then show that $AF = AD + \dfrac{DH \times AC}{AB}$.

Having marked point F, the telescope is made horizontal by being directed to F.

20.* A *pantograph* is a machine for drawing a plane figure similar to a given plane figure, and is useful for enlarging and reducing maps and drawings. It consists of four bars, parallel in pairs and jointed at B, C, D, and E. A turns upon a fixed pivot, and pencils are carried at D and F. BD and DE are so adjusted as to form a parallelogram $BCED$ and such that any required ratio $\dfrac{AB}{AC}$ is equal to $\dfrac{CE}{CF}$.



Show that (1) A, D, and F are always in a straight line, and (2) the ratio $\dfrac{AD}{AF}$ remains constant and equal to the given ratio $\dfrac{AB}{AC}$, so that, if the pencil F traces a given figure, the pencil D will trace a similar figure, the ratio of similitude being the fixed ratio $\dfrac{AD}{AF}$.

Suggestion. — Prove $\triangle ABD \sim \triangle ACF$.

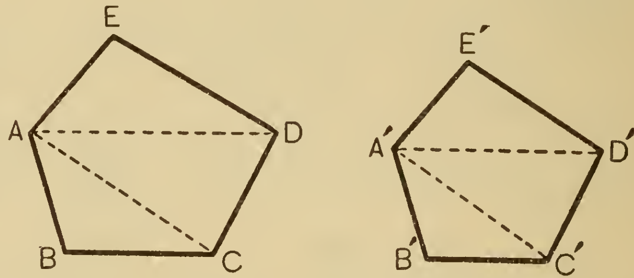
21.* In the pantograph in the preceding problem, show that if the

pencil D traces a straight line, pencil F will trace a straight line also. [See (2) in the preceding problem.]

135. Similar polygons. — Any two polygons that may be decomposed into the same number of triangles, similar each to each and similarly placed, are **similar**.

136. Theorem. — *In any two similar polygons,* (1) *the homologous angles are equal;* (2) *the homologous sides are proportional.*

Suggestions. To prove (1), show that the homologous angles are either the equal angles of similar triangles or the sums of such angles.



To prove (2), use the proportion between the homologous sides of similar triangles, and Axiom I. Thus, $\frac{BC}{B'C'}=\frac{AC}{A'C'}=\frac{CD}{C'D'}=$, etc.

137. Corollary. — *In any two similar polygons the ratio of any two homologous sects equals the ratio of any other two homologous sects.*

(Prove by similar triangles.)

138. Corollary. — *In any two similar polygons, the perimeters have the same ratio as any two homologous sides.*

Proof. 1. By § 136, $\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{DC}{C'D'}=\text{etc.}=r.$

2. Hence, $AB=r\times A'B'$, $BC=r\times B'C'$, $CD=r\times C'D'$, etc.

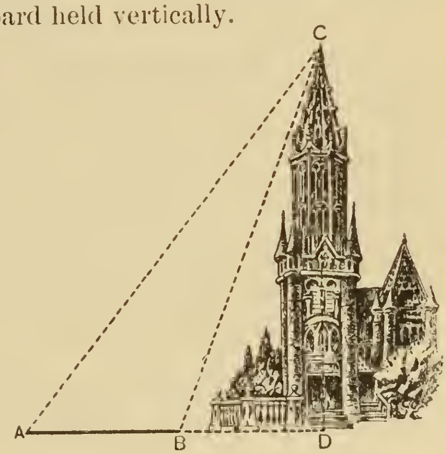
3. $\therefore AB+BC+CD+\cdots=r\times A'B'+r\times B'C+r\times C'D'+\cdots=r(A'B'+B'C'+C'D'+\cdots).$

4. $\therefore \frac{AB+BC+CD+\cdots}{A'B'+B'C'+C'D'+\cdots}=r=\frac{AB}{A'B'}=\text{etc.}$

139.* **A map or plan: drawing to a scale.** — If two figures are similar, one is sometimes called a **map**, or **plan**, of the other. Maps or plans are always *drawn to scale*, *i.e.* in the map or plan the lines are drawn *proportional* to the distances which they represent. Thus, in drawing a map of the United States, in which the distance of 190 mi. in any direction is represented by a sect one inch long, application is made of the fact that *in similar figures, the homologous sects are proportional.*

A distance, such as the height of a mountain, the distance between two mountain peaks, or between two church towers, etc., can be obtained by making certain measurements, then making a map of the measurements, and computing the unknown distance from this map.

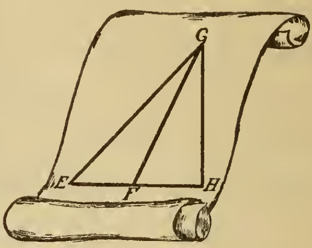
NOTE. — If the school does not have more convenient instruments, a flat board, as a drawing board, and a straightedge with two sights may be used for the purpose of measuring angles. To measure the angles between the directions to two objects in a horizontal plane, pin a piece of paper to the board, and support the board in a horizontal position. Sight the straightedge toward the first object, and draw a line along the straightedge. Then turning the straightedge, without moving the board, in the direction of the other object, draw a second line. In making a map, this paper may then be removed, and an angle constructed equal to the angle made by the two lines. In measuring an angle in a vertical plane, the straightedge may be fastened to the board and the board held vertically.



EXAMPLE. *To find the height CD of some object, such as the spire of a church, without approaching the foot of it.*

Measure off a distance AB toward the foot of the object. At A, measure the angle BAC. At B, measure the angle DBC. Suppose that AB is 50 ft. long. Now, on a piece of paper, draw EF 2 in. long, and extend it. Construct angle FEG equal

to angle BAC. Construct angle HFG equal to angle DBC. The sides of these angles will meet at a point G. From G draw GH perpendicular to EF prolonged. Measure GH. Then the ratio of CD to AB will equal the ratio of GH to EF. From this, CD can be computed.



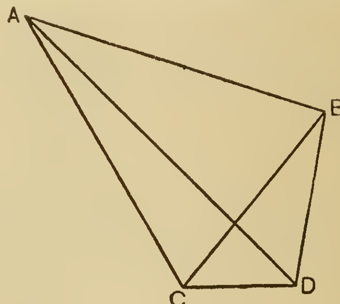
The map is said to be drawn to the *scale of 25 ft. to the inch*, because AB, which was 50 ft., was represented in the map by EF, which was 2 in. Any distance in the problem may be found by multiplying the length in inches of the sect which represents it in the map by 25, and calling the product a number of feet. Thus, if EG is 6 in., AC is 25×6 ft., or 150 ft.

EXERCISES *

1. In the figure of § 139, if GH had been found to be $3\frac{3}{4}$ in., what would CD have been? If EF had been taken 4 in. long, and GH found to be $6\frac{1}{2}$ in., how long would CD have been? If AB had been measured off 100 ft. long, then EF taken 5 in. long, and GH found to be 4 in., how long would CD have been? What are the *scales* in the various suppositions?

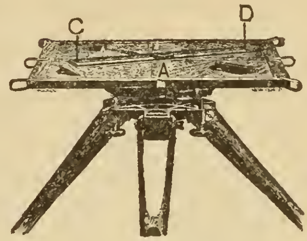
2. The angle of elevation of the top of a flag staff when measured at a point A is 30°. If I walk 80 ft. toward the foot of the staff to B, the angle of elevation is found to be 65°. By using the protractor and drawing to scale, find the height of the staff.

3. A and B are two forts in the lines of the enemy, and it is desired to know their distance apart and their distances from our lines. From point C in our lines, we measure $\angle ACD$ and $\angle BCD$. Then we go to a second point D and measure $\angle CDA$ and $\angle CDB$. $\angle ACD = 120°$, $\angle BCD = 50°$, $\angle CDA = 45°$, $\angle CDB = 100°$, and $CD = 2000$ ft. Draw a plan to the scale of 500 ft. to the inch, and find the distances AC, BD, and AB.



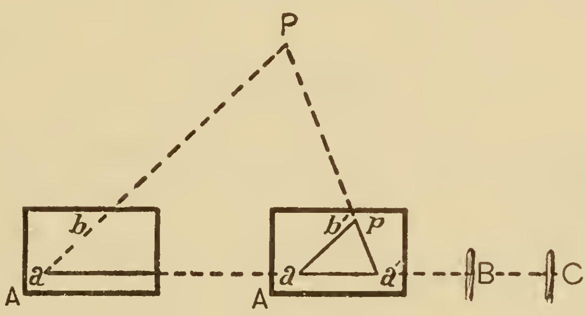
4. Measure heights and distances between objects, such as two church spires, by this method.

140.* **A plane table.** — A *plane table* is used in locating points and laying out maps or plans where great accuracy is not required. It consists of a table or board which may be covered with paper upon which the points and lines may be delineated, a tripod upon which to adjust the table, and a straightedge A to which are fastened two sights C and D.



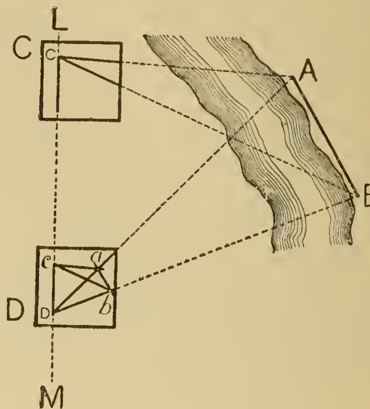
NOTE. — A board 12 by 18 in. makes a convenient size. It should be well seasoned and well jointed to prevent warping. A center of soft white pine to which the paper may be easily fastened, surrounded by a frame of hard wood, makes a good table.

141.* **To use a plane table.** — From some point A take a base line AC by sighting two natural objects or by erecting two poles B and C. Level the table and sight along the straightedge from a (directly over A) toward the object P, and mark on the paper line ab along the straightedge. Place a



pin at a. Now measure along AC any convenient distance to A', so that the angles of the triangle are not too small, say not less than 30°, and represent it by aa' to any scale, on the paper. From a', directly over A', sight the object P and draw on the paper $a'b'$, crossing ab at p. Show that triangles $aa'p$ and $AA'P$ are similar. Show how to find AP or $A'P$ from AA' and aa', whose lengths are known.

142.* The distance between any two inaccessible objects. — Let A and B be the two objects. Let LM be the base line. With the plane table at C draw lines toward A, B, and M. Now measure the distance from C to D, and represent it by any convenient scale upon the base line drawn on the paper of the plane table. Let it be called cd. From d, directly over D, trace lines toward A and B. Show that the figures upon the plane table are similar to those in nature, and hence that AB is represented by the same scale as CD. How then may AB be found? If $\frac{1}{4}$ in. on the plane table represents 1 yd., how far from A to B when the line upon the plane table which represents the distance is $1\frac{3}{4}$ in.?



EXERCISES *

1. By the use of a plane table, find the distances from a point to 5 accessible objects. Test the accuracy of your work by actually measuring to the objects.

Caution. — Use very fine lines and be extremely careful in your measurements of the map or plan. Every error of the map is vastly multiplied in the figure in nature. Thus, if $\frac{1}{8}$ in. represents a rod, an error of $\frac{1}{16}$ in. in the map is an error of $\frac{1}{2}$ rd. in nature.

2. By use of the plane table, find the distance between two objects whose distance may be measured. Check by actually measuring.

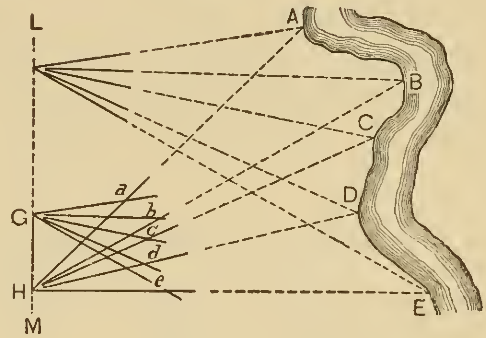
3. Practice until you find that you can make very close estimates by use of the plane table.

4. Find the distance between two points one of which is accessible,

as the distance across a river or lake, if there are two such points in the neighborhood.

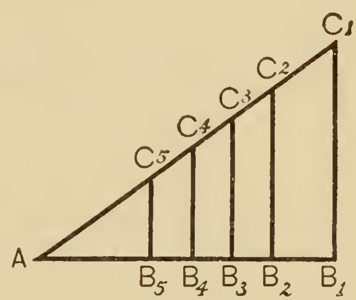
5. Find the distance between two distant objects.

6. By taking your base line in a known direction (east and west, or north and south), find the direction of one object from another and the distance between them. (Use a protractor to measure the angle required.)



7. Use the plane table in constructing a map of the bank of a river or lake, or the location of buildings or trees, or any other convenient thing. Study the figure and tell how it may be done.

143.* Trigonometric ratios. — If one acute angle of a right triangle is known, all of its angles are known. (Why?) Thus, if one angle of a right triangle is 40°, the others are 90° and 50°. Hence, all right triangles with a given acute angle are similar. (Why?)



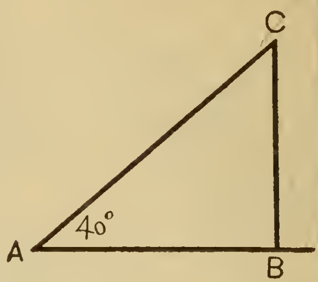
It follows, then, that if from points on one side of an angle lines be dropped perpendicular to the other side, these lines will form with the sides of the given angle a set of similar triangles.

Hence the ratio of any two sides of any right triangle, having a given acute angle, is the same as the ratio between the homologous sides of any other right triangle having the same acute angle. This is expressed by saying:

For any given acute angle of a right triangle, there is a constant ratio between any two sides, whatever their lengths.

These ratios are called **trigonometric ratios.**

144.* Naming trigonometric ratios. — Using a protractor, construct an angle of 40°. From a point C on one side draw a perpendicular to the other side, forming triangle ABC.



Accurately measure CB and AC. Find the ratio of CB to AC, expressed decimally. Keeping the angle 40°, extend the side and make larger triangles. Find the ratio of the altitude to the hypotenuse in each, and take the average of the results. If the work is carefully done, you should find $\frac{CB}{AC} = .6428$, nearly. This is called the **sine** of an angle of 40°. It is written **sin 40° = .6428.**

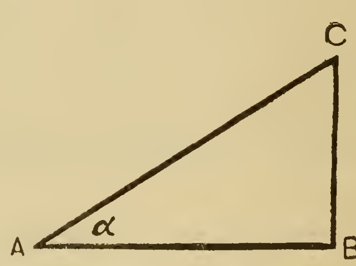
In the same way, find the ratio of AB to AC. If carefully done, you will find $\frac{AB}{AC} = .7660$. This is called the **cosine** of 40°. It is written **cos 40° = .7660.**

In the same way, find the ratio of CB to AB. It will be found to be about .8391. This ratio is called the **tangent** of 40°. It is written **tan 40° = .8391.**

EXERCISE

Accurately construct right triangles with angles of 20°, 30°, 50°, and compute the three trigonometric ratios of each.

145.* Use made of trigonometric ratios. — It must be evident that if any side of a right triangle is known, and the trigonometric ratios of either acute angle are known, the other sides may be found.



1. Thus, since $\frac{BC}{AC} = \sin \alpha$, $CB = AC \sin \alpha$.

2. Since $\frac{AB}{AC} = \cos\alpha$, $AB = AC\cos\alpha$.

3. Since $\frac{CB}{AB} = \tan\alpha$, $CB = AB\tan\alpha$.

146.* Trigonometric tables. — The values of the trigonometric ratios of angles have been accurately computed. The following table shows these values for any integral number of degrees from 0° to 90°:

VALUES OF SINES, COSINES, AND TANGENTS

Deg.	Sine	Cosine	Tangent	Deg.	Sine	Cosine	Tangent
1	.0175	.9998	.0175	21	.3584	.9336	.3839
2	.0349	.9994	.0349	22	.3746	.9272	.4040
3	.0523	.9986	.0524	23	.3907	.9205	.4245
4	.0698	.9976	.0699	24	.4067	.9135	.4452
5	.0872	.9962	.0875	25	.4226	.9063	.4663
6	.1045	.9945	.1051	26	.4384	.8988	.4877
7	.1219	.9925	.1228	27	.4540	.8910	.5095
8	.1392	.9903	.1405	28	.4695	.8829	.5317
9	.1564	.9877	.1584	29	.4848	.8746	.5543
10	.1736	.9848	.1763	30	.5000	.8660	.5774
11	.1908	.9816	.1944	31	.5150	.8572	.6009
12	.2079	.9781	.2126	32	.5299	.8480	.6249
13	.2250	.9744	.2309	33	.5446	.8387	.6494
14	.2419	.9703	.2493	34	.5592	.8290	.6745
15	.2588	.9659	.2679	35	.5736	.8192	.7002
16	.2756	.9613	.2867	36	.5878	.8090	.7265
17	.2924	.9563	.3057	37	.6018	.7986	.7536
18	.3090	.9511	.3249	38	.6157	.7880	.7813
19	.3256	.9455	.3443	39	.6293	.7771	.8098
20	.3420	.9397	.3640	40	.6428	.7660	.8391

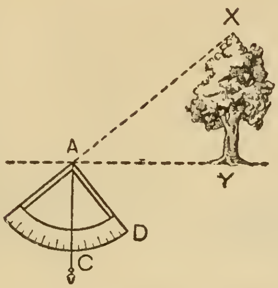
VALUES OF SINES, COSINES, AND TANGENTS (*Continued*)

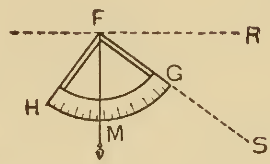
Deg.	Sine	Cosine	Tangent	Deg.	Sine	Cosine	Tangent
41	.6561	.7547	.8693	66	.9135	.4067	2.2460
42	.6691	.7431	.9004	67	.9205	.3907	2.3559
43	.6820	.7314	.9325	68	.9272	.3746	2.4751
44	.6947	.7193	.9657	69	.9336	.3584	2.6051
45	.7071	.7071	1.0000	70	.9397	.3420	2.7475
46	.7193	.6947	1.0355	71	.9455	.3256	2.9042
47	.7314	.6820	1.0724	72	.9511	.3090	3.0777
48	.7431	.6691	1.1106	73	.9563	.2924	3.2709
49	.7547	.6561	1.1504	74	.9613	.2756	3.4874
50	.7660	.6428	1.1918	75	.9659	.2588	3.7321
51	.7771	.6293	1.2349	76	.9703	.2419	4.0108
52	.7880	.6157	1.2799	77	.9744	.2250	4.3315
53	.7986	.6018	1.3270	78	.9781	.2079	4.7046
54	.8090	.5878	1.3764	79	.9816	.1908	5.1446
55	.8192	.5736	1.4281	80	.9848	.1736	5.6713
56	.8290	.5592	1.4826	81	.9877	.1564	6.3138
57	.8387	.5446	1.5399	82	.9903	.1392	7.1154
58	.8480	.5299	1.6003	83	.9925	.1219	8.1443
59	.8572	.5150	1.6643	84	.9945	.1045	9.5144
60	.8660	.5000	1.7321	85	.9962	.0872	11.4301
61	.8746	.4848	1.8040	86	.9976	.0698	14.3006
62	.8829	.4695	1.8807	87	.9986	.0523	19.0811
63	.8910	.4540	1.9626	88	.9994	.0349	28.6363
64	.8988	.4384	2.0503	89	.9998	.0175	57.2900
65	.9063	.4226	2.1445	90	1.0000	.0000	∞

147.* Angles of elevation or depression. — Angles of elevation or depression may be found by the use of a quadrant, divided into degrees, and furnished with a plumb line fixed at the center of the circle and two sights fixed along one of the radii.

QUESTIONS. — **1.** What angle measures the elevation of X? Why?

2. What angle measures the depression of S? Why?





EXERCISES*

1. Construct an angle whose sine is $\frac{3}{4}$. Whose cosine is $\frac{3}{4}$. Whose tangent is $\frac{3}{4}$.

2. Can you construct an angle whose sine is $\frac{5}{4}$? Why?

3. Can you construct an angle whose cosine is $\frac{5}{4}$? Whose tangent is $\frac{5}{4}$?

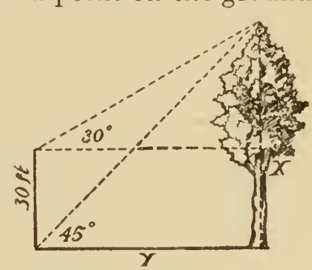
4. Construct an angle whose tangent is 2. Can you construct an angle whose sine or cosine is 2? Why?

5. The angle of elevation of a steeple 120 ft. distant is 30°. Find its height.

6. The distance from the base to the top of a hill, up a uniform incline of 40°, is 300 yd. What is the altitude of the top above the base?

7. A kite string 1000 ft. long makes an angle of 60° with horizontal. How high is the kite, not counting the sag in the string? It is directly over a spot how far from the holder of the string?

8. The angle of elevation of a tree is 45° from a point on the ground and in the same horizontal plane as the tree. From an upper window, 30 ft. from the ground, the angle of elevation is but 30°. How high is the tree?

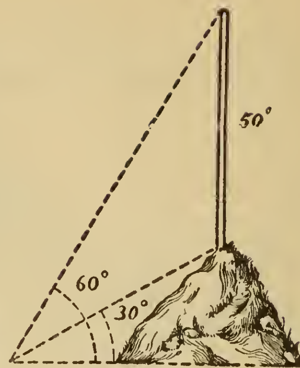


SUGGESTION. — Let x be the required height. We have two equations.

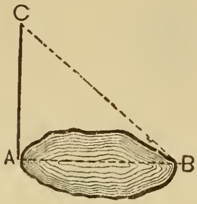
$$x - 30 = y \tan 30°.$$
$$x = y \tan 45°.$$

Solve for x.

9. A 50-ft. pole stands on the top of a mound. The angles of elevation of the top and bottom of the pole are respectively 60° and 30°. Find the height of the mound.

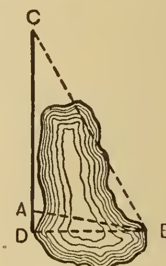


10. To find the distance across a lake from a point A to a point B, a man measured 100 rd. to a point C on a line perpendicular to the line AB and found that the angle ACB was 50°. Find the distance across the lake.

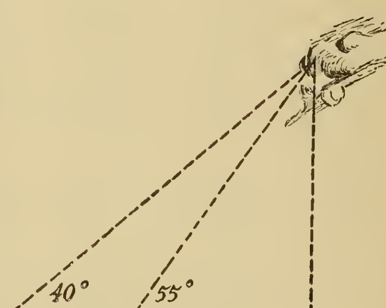


11. To find the distance from A to B, across a lake, being unable to measure perpendicularly to AB, I measured 120 yd. to C, making $\angle CAB = 100°$, and observed that $\angle ACB = 30°$. Find the distance from A to B.

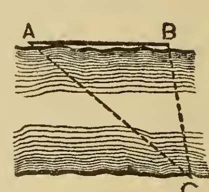
SUGGESTION. — Find AD, or DB. Then from this find AB.



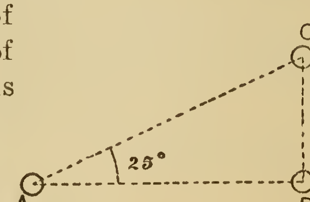
12. How high is a cloud when two observers, so placed as to both be in a vertical plane with the cloud, and 880 yd. apart, observe the angles of elevation to be 40° and 55°, respectively?



13. A tree is standing on a bluff on the opposite side of a river from the observer. Its foot is at an elevation of 45° and its top at 60°. Compare the height of the bluff with that of the tree (*i.e.* find the ratio). What measurement would you use to find the height of the tree? the height of the bluff? the width of the river?



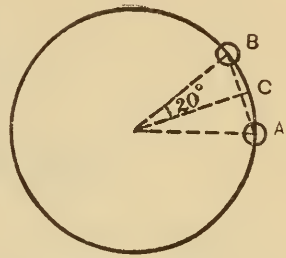
14. A base line AB 300 ft. long was measured along the bank of a river. From one end of the line, a tree C, on the opposite bank, made an angle of 40°. From the other end it made an angle of 80° with the extension of the line through this point. Find the width of the river.



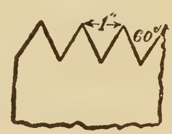
15. Draughtsmen often make drawings with the angles marked in degrees, and leave it for

the machinist to figure out the dimensions in the drawings. Thus, a drawing shows three holes, A, B, C, to be bored in a metal plate, with $AB = 3''$ and $\angle A = 25°$. Compute the distances from A to C and from B to C. $(\angle B = 90°.)$

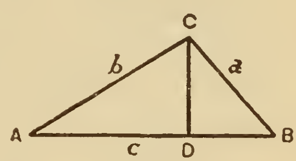
16. A machinist is required to bore nine holes in a circle with radius 7 in., the holes to be at equal intervals. He must find the distance AB between two consecutive holes in order to locate them with the dividers. He divides 360° by 9, getting 40°. Hence, using 20°, he computes AC, and likewise CB, then adds. Compute AB.



17. On an iron bolt the threads are cut ten to the inch (pitch = .1 in.). The threads are the "standard V" threads, in which the angle of the groove is 60°. What is the depth of the groove? If the outside diameter of the thread is $\frac{5}{8}$ in., what is the "root diameter" (diameter measured at bottom of groove)?



18. In $\triangle ABC$, show that the altitude $CD = b \sin A = a \sin B$.



19. Show from the results of Ex. 18 that $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$.

20. By drawing another altitude in the triangle ABC, Ex. 18, show that $b \sin C = c \sin B$, and thus show that in any triangle

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

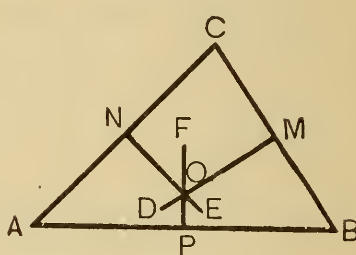
This important relation is known as **the law of sines.**

CHAPTER VI

CONCURRENT LINES OF A TRIANGLE

148. If three or more straight lines meet in a point, they are **concurrent** at that point.

149. Theorem. — *In any triangle, the perpendiculars erected at the middle points of the sides are concurrent at a point equally distant from the three vertices.*



Hypothesis. In $\triangle ABC$, NE, MD, and PF are the perpendicular bisectors of sides AC, BC, and AB, respectively.

Conclusion. NE, MD, and PF are concurrent at a point equally distant from A, B, and C.

Proof. 1. Since MD and NE are not parallel, (Why?)

2. $\therefore$ they will meet, say at O.

3. Now O is equidistant from A and C; also from B and C.

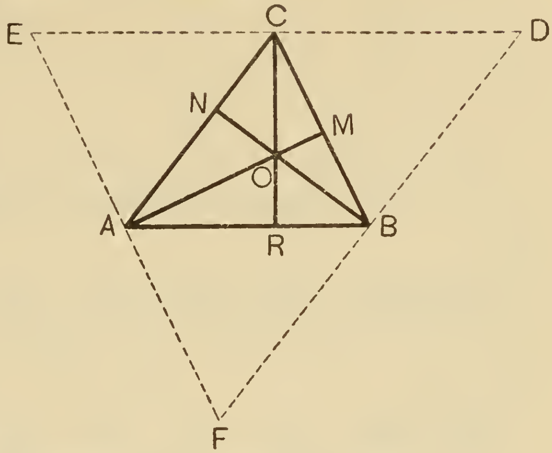
4. $\therefore$ O is equidistant from A and B, and hence on PF; that is, PF passes through O.

150. Theorem. — *In any triangle, the three altitudes are concurrent.*

Hypothesis. AM, BN, and CR are altitudes of $\triangle ABC$.

Conclusion. AM, BN, and CR are concurrent.

Proof. 1. Draw through each vertex a straight line parallel to the opposite side, forming $\triangle DEF$.



2. The altitudes AM, BN, and CR, of $\triangle ABC$ will be perpendicular to sides of $\triangle DEF$ at A, B, and C.

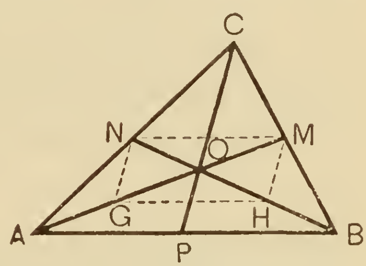
3. Also $CABD$ and $EABC$ are ▱s.

4. $\therefore$ $EC = AB = CD$; *i.e.* C is the middle point of ED.

Likewise, B and A are middle points of FD and FE, respectively.

5. $\therefore$ altitudes AM, BN, and CR are concurrent (§ 149).

151. Theorem. — *The medians of any triangle are concurrent in a point of trisection of each.*



Hypothesis. AM, BN, and CP are medians of $\triangle ABC$.

Conclusion. AM, BN, and CP are concurrent at a point of trisection of each.

Proof. 1. Since AM and BN are not parallel, (Why?) let them intersect in O.

2. Let G and H be the middle points of OA and OB, respectively.

3. Draw GH, HM, MN, and NG.

4. Now both MN and GH are parallel to AB and equal to $\frac{1}{2}$ AB.

5. $\therefore$ $MN = GH$, and $MN \parallel GH$.

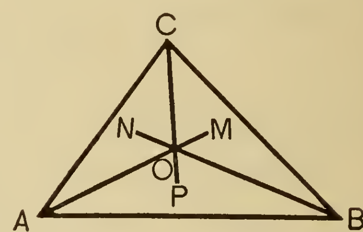
6. $\therefore$ $GHMN$ is a ▱.

7. $\therefore$ $OM = OG = GA$ and $NO = OH = HB$.

8. $\therefore$ the median AM passes through a point of trisection of BN; also BN passes through a point of trisection of AM.

By similar reasoning, CP passes through the trisection point of both AM and BN, and hence through O, and O is a point of trisection of CP.

152. Theorem. — *The bisectors of the three angles of any triangle are concurrent in a point equally distant from the three sides.*



Hypothesis. AM, BN, and CP bisect $\angle A$, $\angle B$, and $\angle C$, respectively.

Conclusion. AM, BN, and CP are concurrent in a point equally distant from AB, AC, and BC.

Suggestions. 1. The nature of the theorem suggests a method of proof similar to that of §§ 149 and 151.

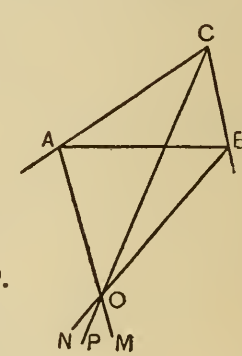
2. Will AM and BN meet at some point O?

3. Show the relation of O to sides AB and AC; to BA and BC; and hence to AC and BC.

4. Show then that O is on CP.

EXERCISES

1. In any triangle, the bisectors of an interior angle and the two opposite exterior angles are concurrent.



Suggestions. — 1. Show that AM and BN, the bisectors of the exterior angles at A and B, respectively, meet at a point O.

2. Show that O is on PC, the bisector of $\angle C$.

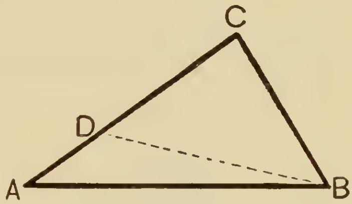
2. Prove the theorem of § 151 by the following construction:

Show that the medians AM and BN meet at O. Then draw CP through O, and produce it to Q so that $CO = OQ$. Draw AQ and BQ, and show that $AOBQ$ is a parallelogram, and hence that $AP = PB$, etc.

CHAPTER VII

THEOREMS OF INEQUALITY

153. Theorem. — *If two of the sides of a triangle are unequal, the angles opposite are unequal, and the larger angle is opposite the longer side.*



Hypothesis. In $\triangle ABC$, $AC > BC$.

Conclusion. $\angle B > \angle A$.

Proof. 1. Lay off on AC, $CD = BC$, and draw BD.

2. $\because AC > BC$, D will fall between A and C, and BD will fall within $\triangle ABC$.

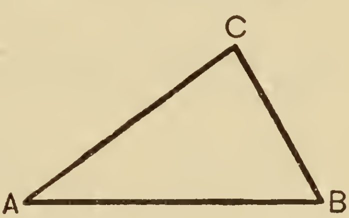
3. $\therefore \angle ABC > \angle DBC$.

4. Also $\angle CDB > \angle A$, and $\angle CDB = \angle DBC$.

5. $\therefore \angle DBC > \angle A$.

6. $\therefore \angle ABC > \angle A$.

154. Theorem. — *If two of the angles of a triangle are unequal, the sides opposite are unequal, and the longer side is opposite the larger angle.*



Hypothesis. In $\triangle ABC$, $\angle B > \angle A$.

Conclusion. $AC > BC$.

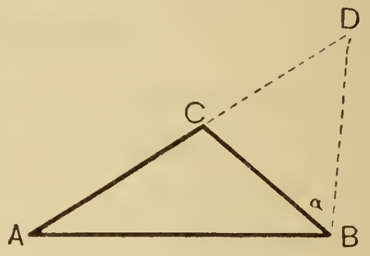
Suggestions. 1. AC must be equal to, smaller than, or greater than BC. Hence prove that $AC \neq BC$, and $AC \not< BC$.

2. Suppose $AC = BC$, and compare the angles B and A. (How?)

3. Suppose $AC < BC$, and compare the angles B and A by § 153.

4. Since $AC \neq BC$ and $AC \not< BC$, what must follow?

155. Theorem. — *In any triangle, any side is less than the sum of the other two sides and greater than their difference.*



Hypothesis. In $\triangle ABC$, AB is the longest and BC the shortest side.

Conclusion. $AB < AC + BC$, and $BC > AB - AC$.

Proof. 1. Produce AC to D, making $CD = BC$, and draw BD.

2. $\angle D = a$.

3. $\therefore \angle ABD > \angle D$.

4. $\therefore AD > AB$; *i.e.* $AC + CD > AB$.

5. But $CD = BC$.

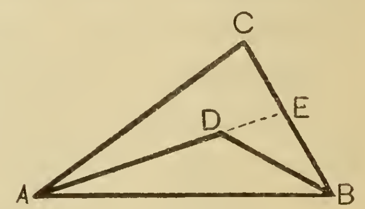
6. $\therefore AC + BC > AB$; or $AB < AC + BC$.

7. $\therefore BC > AB - AC$.

Why, under the supposition that AB is the longest, and BC the shortest side, are the conclusions in steps 6 and 7 sufficient to prove the theorem without considering other cases?

Note. — Here we really prove the assumption of § 8. III.

156. Theorem. — *In any triangle, the sum of two sects drawn from a point within the triangle to the extremities of any side is less than the sum of the other two sides.*



Hypothesis. In $\triangle ABC$, the sects AD and BD are drawn from point D within $\triangle ABC$ to A and B.

Conclusion. $AD + BD < AC + BC$.

Proof. 1. Produce AD to intersect BC in E.

2. Now $AC + CE > AE$.

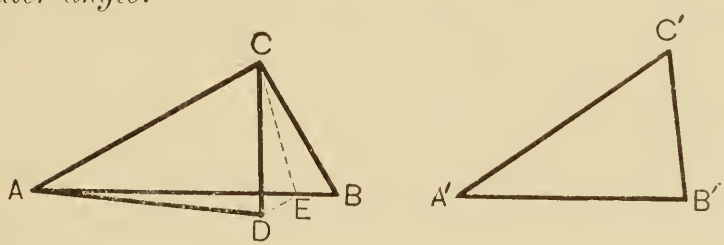
3. $\therefore AC + CE + EB > AE + EB$; *i.e.* $AC + BC > AE + EB$.

4. Also $EB + ED > BD$.

5. $\therefore EB + ED + AD > BD + AD$; *i.e.* $AE + EB > BD + AD$.

6. $\therefore AC + BC > AD + BD$.

157. Theorem. — *If two triangles have two sides of one equal to two sides of the other, but the included angle of the first greater than the included angle of the second, the sides opposite these angles are unequal, and the greater side lies opposite the greater angle.*



Hypothesis. $\triangle ABC$ and $A'B'C'$ have $AC = A'C'$, $BC = B'C'$, and $\angle C > \angle C'$.

Conclusion. $AB > A'B'$.

Proof. 1. Place $\triangle A'B'C'$ on $\triangle ABC$ so that $A'C'$ will coincide with AC, the sides $B'C'$ and $A'B'$ taking the positions DC and AD, respectively.

2. Bisect $\angle BCD$, and let the bisector CE cut AB in E. Draw DE.

3. Now $\triangle DEC \cong \triangle EBC$. (Why?)

4. $\therefore DE = EB$.

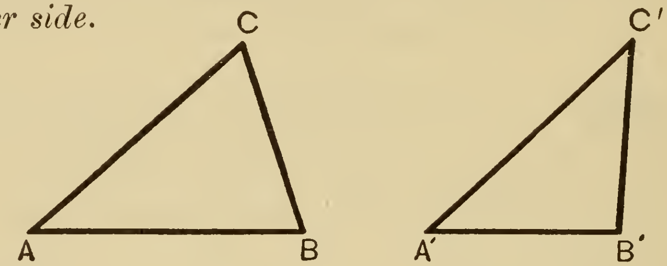
5. Now $AE + DE > AD$.

6. $\therefore AE + EB > AD$; *i.e.* $AB > A'B'$.

Might D fall upon AB? What is the proof in that case?

Would the above method of proof apply if D fell within $\triangle ABC$?

158. Theorem. — *If two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, the angles opposite these sides are unequal, and the greater angle lies opposite the greater side.*



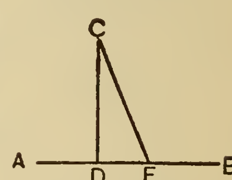
Hypothesis. In $\triangle\triangle\, ABC$ and $A'B'C'$, $AC = A'C'$, $BC = B'C'$, and $AB > A'B'$.

Conclusion. $\angle C > \angle C'$.

Suggestions. 1. Either $\angle C > \angle C'$, $\angle C = \angle C'$, or $\angle C < \angle C'$.
2. Show that $\angle C \neq \angle C'$ and $\angle C \not< \angle C'$.

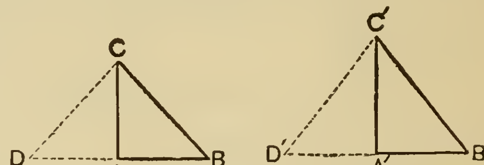
EXERCISES

1. Prove that the shortest distance from a point to a line is the length of the perpendicular sect drawn from the point to the line. (This was assumed earlier.)



SUGGESTIONS. — Let CD be $\perp AB$ and CE not $\perp AB$. Show $CD < CE$ by comparing $\angle\angle\, D$ and E.

2. Prove that if two right triangles have their hypotenuses equal and an acute angle of one greater than the corresponding acute angle of the other, the legs opposite these angles are unequal, and the longer leg is opposite the larger angle.



SUGGESTIONS. — If $BC = B'C'$, and $\angle C > \angle C'$, produce AB through A to D, making $AD = AB$, and draw CD; also produce $A'B'$ through A' to D', making $A'D' = A'B'$, and draw $C'D'$. Now show that $DB > D'B'$, and hence $AB > A'B'$.

3. State and prove the converse of Exercise 2.

4. If two sides of a triangle are unequal, the median to the third side makes the larger angle with the shorter of the unequal sides.

SUGGESTIONS. — Draw a triangle ABC with $AC > CB$ and let CD be the median. Prove that $\angle DCB > \angle ACD$. Produce CD to E, making $CD = DE$; and draw AE. Compare $\angle DCB$ with $\angle E$, and $\angle ACD$ with $\angle E$.

5. Using the figure drawn for Exercise 4, prove that $CD < \frac{1}{2}(AC + CB)$.

6. Using the figure drawn for Exercise 4, prove that $CD > \frac{1}{2}(AC + CB - AB)$.

SUGGESTION. — Compare CD with $AC - AD$, then with $CB - DB$.

159. Method of Attack. — From the proofs and suggestions of the book, it must be evident that there is a definite *method of attack* in studying theorems. Up to this point all proofs have depended upon one or more of three fundamental operations — a comparison of sects, angles, or ratios. In attacking a theorem, the student should have clearly in mind all theorems that relate to the *equality* or *inequality* of sects, angles, and ratios. Thus,

Sects are proved equal by showing that they are:

(1) *homologous parts of congruent triangles;*

(2) *opposite sides of a parallelogram;*

(3) *sides opposite equal angles in a triangle;*

(4) *parallel sects comprehended between parallel lines; etc.*

Angles are proved equal by showing that they are:

(1) *homologous parts of congruent triangles;*

(2) *complements or supplements of equal angles;*

(3) *vertical angles;*

(4) *angles opposite equal sides of a triangle;*

(5) *certain angles made by parallel lines cut by a transversal;*

(6) *angles whose sides are parallel or perpendicular to each other; etc.*

The steps taken in dealing with a theorem may be summarized as follows:

I. *If the theorem is to be true, the figure should be accurately drawn so that the theorem looks true in the figure.*

II. *Get clearly in mind the hypothesis and the conclusion, being sure to understand the meanings of all terms involved.*

III. *Recall all theorems or definitions that relate to the thing to be proved — the equality or inequality of sects, angles, or ratios.*

IV. *Having determined from the figure upon what theorem or definition the proof depends, seek to see if such theorem or definition applies at once to the figure. If not, try to discover by the same process upon what third theorem or definition this one depends. Continue in this way until, if possible, some known theorem or definition is seen to apply.*

V. *If none of the known theorems or definitions apply to the figure as given, try to draw auxiliary lines that will involve the elements wanted, and that will give a figure to which some of the known theorems or definitions will apply.*

VI. *If successful, apply the known theorem or definition, and give the proof.*

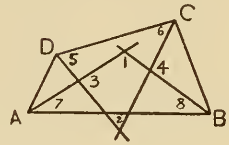
Steps I–V constitute the **analysis**, and step VI the **direct** or **synthetic proof** of the theorem.

VII. *If no direct proof can be found, assume the theorem false, and show that the assumption leads to an absurdity or contradicts a known truth.*

Converse theorems are generally more easily proved by step VII, called the **indirect method**.

MISCELLANEOUS EXERCISES

1. The bisectors of the angles of any quadrilateral form a quadrilateral whose opposite angles are supplementary. Discuss also the special cases when the given quadrilateral is a parallelogram; a rectangle; a square.



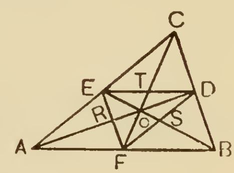
Analysis. — 1. $\angle 1 + \angle 2 =$ st. $\angle$, if $\angle 5 + \angle 6 + \angle 7 + \angle 8 =$ st. $\angle$.

2. $\angle 5 + \angle 6 + \angle 7 + \angle 8 =$ st. $\angle$, if $\angle A + \angle B + \angle C + \angle D = 2$ st. $\measuredangle$s.

3. But $\angle A + \angle B + \angle C + \angle D = 2$ st. $\measuredangle$s.

4. Hence, begin by proving that $\angle A + \angle B + \angle C + \angle D = 2$ st. $\measuredangle$s.

2. If AD, BE, and CF are the medians of $\triangle ABC$, and intersect at O, then O is the intersection of the medians of $\triangle DEF$ also.



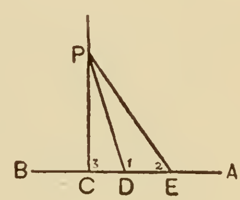
Analysis. — 1. O is the intersection of the medians of $\triangle DEF$, if $ET = TD$, $FR = RE$, and $FS = SD$.

2. $ET = TD$, if $ECDF$ is a $\square$.

3. $ECDF$ is a $\square$, if $FD \parallel AC$ and $FD = \frac{1}{2} AC$, which can be proved.

4. Hence, begin by proving $FD \parallel AC$ and $FD = \frac{1}{2} AC$.

3. Of two oblique sects drawn to a line from any point in a perpendicular to the line, the one meeting the given line at the greater distance from the foot of the perpendicular is the greater. That is, if $PC \perp AB$, and $CE > CD$, prove $PE > PD$.



Analysis. — 1. $PE > PD$, if $\angle 1 > \angle 2$.

2. $\angle 1 > \angle 2$, if $\angle 1 > \angle 3$ and $\angle 2 < \angle 3$, which can be proved.

3. Hence, begin by proving $\angle 1 > \angle 3$ and $\angle 2 < \angle 3$.

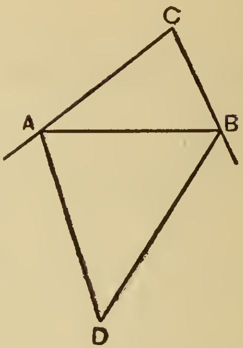
Prove for the case where PE and PD are on opposite sides of PC.

4. Prove the converse of Exercise 3; that is, if $PE > PD$, prove that $CE > CD$.

Analysis. — A study of the figure will show that no easy *direct* proof can be found. Hence begin by assuming that $CE \not> CD$, and hence that either $CE = CD$ or $CE < CD$. Show that either of these suppositions leads to a contradiction of a known truth.

5. The bisectors of the angles of a parallelogram form a rectangle.

6. If the exterior angles at A and B of $\triangle ABC$ are bisected by AD and BD, respectively, $\angle ADB = 90° - \frac{1}{2}\angle C$.

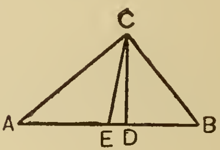


7. If a diagonal of a quadrilateral bisects two of its angles, it is perpendicular to the other diagonal.

8. The bisectors of the angles of a rectangle form a square.

9. The sum of the sects drawn from any point within a triangle to the three vertices is less than the sum of the three sides.

10. The sum of the three sects drawn from any point within a triangle to the three vertices is greater than the half-sum of the three sides.

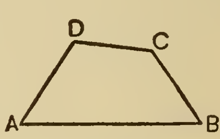


11. In the right triangle ABC, AB is the hypotenuse, $CD \perp AB$, and CE is the median. Prove $\angle DCE = \angle B - \angle A$.

12. The sects joining the middle points of the adjacent sides of any quadrilateral form a parallelogram.

13. The sects joining the middle points of the opposite sides of a quadrilateral bisect each other.

14. The parallelogram (Ex. 12) formed by joining the adjacent sides of a quadrilateral has for its center the middle point of the sect joining the middle points of the diagonals of the quadrilateral.



15. In a quadrilateral $ABCD$, AB is the longest side and CD the shortest. Prove $\angle C > \angle A$, and $\angle D > \angle B$.

16. If straight lines be drawn bisecting the angles of any parallelogram,

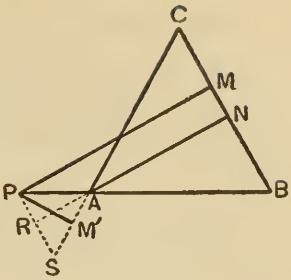
(1) each bisector cuts off an isosceles triangle from the parallelogram;

(2) the sects of the bisectors of two adjacent angles comprehended by the sides of the parallelogram bisect each other;

(3) the diagonals of the rectangle (Ex. 5) formed by the bisectors are parallel to the sides of the parallelogram.

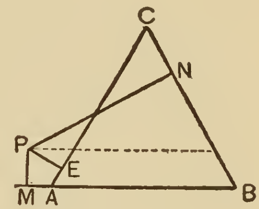
17. If from any point P in the base of an isosceles triangle perpendiculars are drawn to the equal sides, the sum of their lengths is constant for all positions of the point and is equal to the perpendicular from an extremity of the base to the opposite side.

Suggestion. — From P draw an auxiliary line parallel to a side.



18. If, in Exercise 17, the point is taken on the base produced, the *difference* of the perpendiculars to the sides is equal to the perpendicular from an extremity of the base to the opposite side.

19. From any point within an equilateral triangle the sum of the perpendiculars to the sides is equal to an altitude.

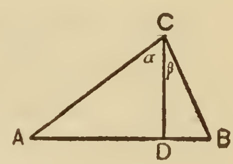


SUGGESTION. — Draw through the point P a line parallel to one side, and use Ex. 17.

20. If the point P in Exercise 19 is moved without the triangle, state and prove the theorem that applies.

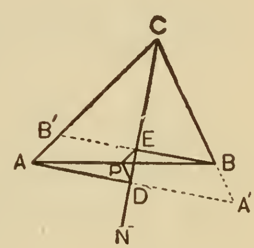
21. The bisectors of the angles of a trapezoid form a quadrilateral two of whose angles are right angles.

22. If through a point midway between two parallel lines two transversals are drawn, they cut off equal sects from the parallel lines.



23. In any isosceles trapezoid the opposite angles are supplementary.

24. The perpendicular from the vertex of any triangle to the base divides the angle at the vertex into two angles whose difference equals the difference of the base angles. $(\alpha - \beta = \angle B - \angle A)$.



25. In $\triangle ABC$, CN bisects $\angle C$. If perpendiculars AD and BE are drawn to CN, and if P is the middle point of AB, then $PD = PE = \frac{1}{2}(AC - BC)$.

SUGGESTION. — Produce CB, and let AD produced meet it at A'. Also produce BE to cut AC at B'.

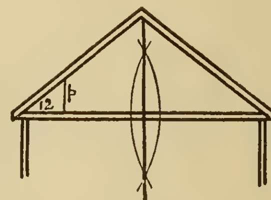
26. In the figure of Exercise 25, prove that $\angle PAD = \angle PBE = \frac{1}{2}(\angle B - \angle A)$.

27. The perpendiculars from the extremities of the base of any triangle upon the median to the base are equal.

28. The locus of points equidistant from two intersecting straight lines is two straight lines perpendicular to each other.

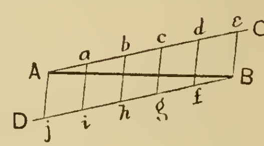
29. What is the locus of points equidistant from two parallel lines?

30. From a variable point on the base of a triangle parallels to the sides are drawn. Show that the locus of the centers of the parallelograms thus formed is a straight line parallel to the base.



31. The figure shows a method used in architecture of drawing a symmetrical roof with a pitch of p inches per foot. Explain the construction, and make one similar to it.

32. Draughtsmen and designers sometimes divide a given sect into any required number of equal parts by the following method: To divide AB into 5 equal parts, draw AC at any convenient angle with AB. Draw BD parallel to AC. Beginning at A, mark off on AC 5 equal sects, Aa, ab, bc, etc., of any convenient length. Beginning at B, mark off on BD 5 sects equal in length to those on AC, Bf, fg, gh, etc. Join e to B, d to f, c to g, etc. These lines divide AB into 5 equal sects. Prove it.



33. The height x of an object may be found without instruments for measuring angles as follows: A person of height h places a staff of known length l vertically at a point B, and finds a position A from which the top of the staff and the top of the object are in line with his eye. The distance a from A to B is measured. He then recedes from the object to a new position C when the staff is removed to D, and repeats the measurement. Make a drawing. Represent the distance from C to D by b, and the distance from D to B by c. From similar triangles, show that

$$(1)\ \frac{x-h}{a+d}=\frac{l-h}{a} \text{ and } (2)\ \frac{x-h}{d+c+b}=\frac{l-h}{b}.$$

Eliminate d from these equations, and solve the resulting equation for x.

34. An interior angle of an equi-angular polygon is $\frac{4}{3}$ of a right angle. How many sides has the polygon?

35. How many sides has a polygon the sum of whose interior angles is equal to the sum of its exterior angles? The sum of whose interior angles is equal to twice the sum of its exterior angles? The sum of whose interior angles is equal to three times the sum of its exterior angles?

CHAPTER VIII

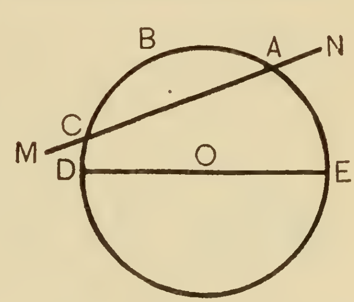
CIRCLES

160. A circle. — The locus of all points in a plane equally distant from a fixed point in the plane is a **circle.** The fixed point is the **center** of the circle. A circle completely encloses a portion of the plane.

NOTE. — The locus which is here called a *circle* is often called a *circumference* of a circle, by writers of geometry. But since no confusion can result in the use of the term, we shall conform in this book to the modern usage in higher mathematics as well as in applied mathematics, and use the term *circle* to denote the locus itself.

161. The lines of a circle. — A straight line cutting a circle is a **secant.** The part of the secant within the circle and intercepted by the circle is a **chord.** If the chord passes through the center it is a **diameter.** The part of a diameter from the center to the circle is a **radius.**

162. The arcs of a circle. — The two parts into which a circle is divided by a chord are called **arcs**, but, unless otherwise stated, the *smaller* of the two arcs is the one considered as corresponding to any chord. Thus, the arc ABC is the arc corresponding to chord AC. The arc is said to be **subtended** by the chord.



163. Corollaries. — From the definitions it follows that

1. *A diameter is equal in length to the sum of two radii.*

2. *The distance of any point within the circle from the center is less than a radius.*

3. *The distance of any point without the circle from the center is greater than a radius.*

4. *All radii of the same circle are equal.*

5. *All diameters of the same circle are equal.*

164. Assumption. — *A circle of given radius may be described about a given point as center.*

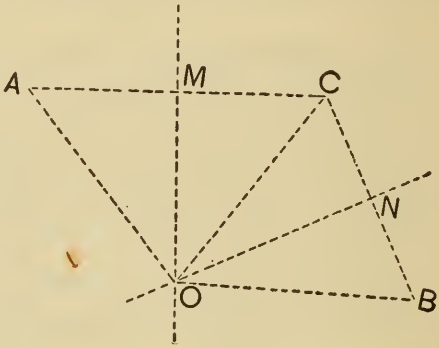
165. Theorem. — *A circle is determined by*

(1) *the center and radius,*

(2) *a diameter,* or

(3) *three points not in the same straight line.*

Since a figure is *determined* by certain elements when one and only one can be drawn with these elements, it can be seen that (1) and (2) follow directly from the definitions. Hence it remains to prove (3).



Hypothesis. A, B, and C are three points not in one straight line.

Conclusion. A, B, and C determine a circle.

Suggestions. 1. Draw AC and BC. Erect perpendicular bisectors to AC and BC. These perpendiculars will meet in some point O. (Why ?)

2. Draw OA, OC, and OB, and prove them equal.

3. Then O is equidistant from A, B, and C.

4. What is the center of the circle? What is the radius?

5. Now show that any circle through A, B, and C must have its center at O.

6. Why is the circle determined?

166. Corollary 1. — *Two circles are congruent if the radius of one is equal to the radius of the other.*

167. Corollary 2. — Conversely, *if two circles are congruent, the radius of one is equal to the radius of the other.*

168. Corollary 3. — *The perpendicular bisector of a chord passes through the center of the circle.*

169. Assumption. — It is evident that

A straight line cannot intersect a circle in more than two points.

170. Constructions. — A circle cannot be drawn until the center and radius are known. To find the center requires locating a point. *A point is determined by the intersection of two loci.* Thus, if a point is known to be upon two straight lines, it must be at their point of intersection. This is illustrated in finding the center of the circle in the following problem.

171. Construction. — *Draw an arc congruent to a given arc.*

Suggestions. 1. First, select any three points upon the arc and find the center of the circle that will pass through them by the theorem of § 165. Upon what does the center lie?

2. Then get the radius and draw the required arc.

EXERCISES

1. Complete the circle of which AB is an arc. How will you find the center? The radius?

2. From the measurements of a piece of a broken wheel, a new wheel is to be cast of the same size. Show how to find the radius of the new wheel.

3. Draw a circle to pass through two given points and have its center on a given line. How will you find the center? The radius? When will the construction be impossible?

4. Draw a circle of given radius to pass through two given points. When is the construction impossible?

5. Pirates buried treasure 75 ft. from a certain tree, and 100 ft. from a straight path which passes the tree at a short distance. Show how to locate the treasure. When are there two possible locations? When but one? When none?

6. If the treasure had been buried equidistant from a certain oak and a certain chestnut and 30 yd. from a certain straight road, show how to locate it.

7. § 165, (3) suggests a method of circumscribing a circle about a triangle, *i.e.* drawing a circle through the three vertices. Describe the method, and circumscribe a circle about a given triangle.

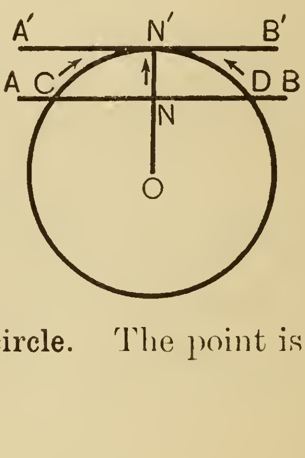
8. Study the construction of the accompanying ornamental drawing, and make a similar one of larger size.

Suggestion. — Draw an equilateral triangle, and construct the drawing about it.

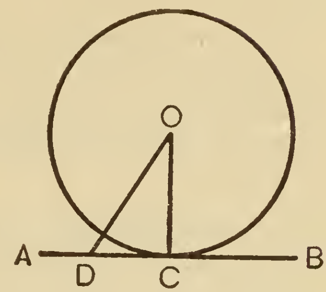
9. If two perpendicular chords bisect each other, each must be a diameter.

10. How many circles can be made to pass through four points in a plane, no three points being in the same straight line, if each circle passes through three and only three points?

172. A tangent. — It is clear from the figure that as the length of the perpendicular ON to the secant AB increases and approaches the length of the radius ON', the points C and D where the secant intersects the circle approach each other. In the limit, where $ON = ON'$, C and D coincide at N'. The line in the position $A'B'$ which *cuts* the circle in *two coincident points*, or, as it is expressed often, *touches* the circle in *one distinct point*, is called a **tangent of the circle.** The point is the **point of tangency.**



173. Theorem. — *A straight line perpendicular to a radius at the point of intersection with the circle is a tangent.*



Hypothesis. OC is a radius of the given circle, and AB is perpendicular to OC at C, the point of intersection.

Conclusion. AB is tangent to the circle.

Proof. 1. Let D be any point on AB other than C. Draw OD.

2. Then, $OD > OC$; *i.e.* greater than a radius. Why?

3. $\therefore$ D lies without the circle. That is, any other point of AB than C lies without the circle. Why?

4. $\therefore$ AB touches the circle in one and but one distinct point, and hence is a tangent.

174. Theorem. — (Converse of theorem, § 173.) *A tangent is perpendicular to the radius drawn to the point of tangency.*

Hypothesis. AB (Fig., § 173) is tangent to the circle at C and OC a radius.

Conclusion. $OC \perp AB$.

Suggestions. 1. Let D be any point on AB other than C. Draw OD.

2. Then D must lie without the circle.

3. Prove $OC < OD$; *i.e.* that OC is the shortest distance from O to AB. See § 29.

175. Construction. — The theorem in § 173 suggests a method of drawing a tangent to a circle from *a point on the circle.* Describe the method, and draw a tangent to a circle from a point on the circle.

Suggestion. This requires finding the center of the circle (How?), drawing a radius to the given point, and erecting a perpendicular.

EXERCISES

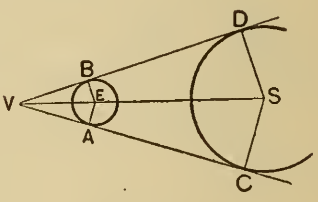
1. The tangents to a circle at the extremities of any diameter are parallel.

2. The perpendicular to a tangent at the point of contact passes through the center of the circle.

SUGGESTION. — Use the indirect proof. Draw the radius to the point of contact. How many perpendiculars can be drawn to a given line at a given point?

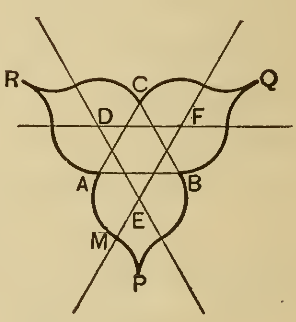
3. A perpendicular from the center of a circle to a tangent passes through the point of contact.

SUGGESTION. — Draw the radius to the point of contact.

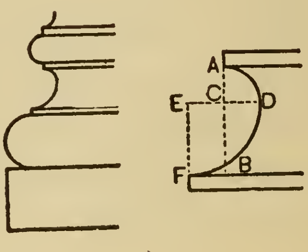


4. Since the earth is smaller than the sun, it casts a conical shadow in space (umbra), from within which one can see no portion of the sun's disk. If S is the center of the sun, E is the center of the earth, and V the end or vertex of the shadow, prove that the length of the shadow, $VE, = \dfrac{ES \times EB}{SD - EB}$.

Approximately, $ES = 92{,}000{,}000$ miles, $SD = 433{,}000$ miles, and $EB = 4000$ miles. Compute VE.



5. Figures such as this, made up of circular arcs, are encountered by students of art and designing. Draw the figure, using for each arc the radius $\frac{3}{4}$ inch. First draw equilateral triangle ABC with each side $1\frac{3}{8}$ inches. Then with A, B, C as centers, and radii $\frac{3}{4}$ inch, draw arcs meeting at D, E, F. Show that the arcs AM and MP, forming the curve AMP, have the same tangent line at M, and hence form one smooth curve, called a "reverse curve."



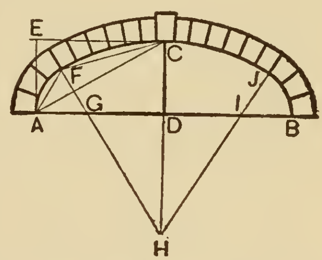
6. Moldings are prominent architectural features. The form of molding varies as the style of architecture. The Greeks and Romans used several forms. Roman moldings are formed from arcs of circles, while Greek moldings are usually formed from conic sections, and hence are more graceful.

The figures show the base of a Roman Ionic column, and the part of it called the *scotia*, enlarged.

Given the height AB of the scotia, to draw it, trisect AB, and from C as center draw arc AD. Make $EC = AC$, and perpendicular to AB. From E as center and radius ED, draw arc DF.

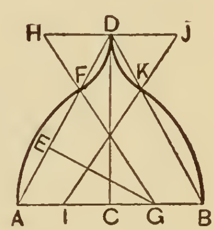
Let the student draw such a scotia. Show that the arcs AD and DF have the same tangent line at D, and hence form one smooth curve, called a "compound curve."

7.* Arches in buildings and bridges are of several different kinds. This figure shows one whose curve is a compound curve formed by the arcs of three circles.



Let AB be the span, and CD, perpendicular bisector of AB, be the rise of the arch. Complete the rectangle $ADCE$. Draw AC. Draw bisectors of $\angle CAE$ and $\angle ECA$, meeting at F. Draw $FG \perp AC$, and extend it to meet CD prolonged at H. Make $DI = DG$. Draw HI, and prolong it. With center G and radius AG, draw arc AF. With center I and radius IB, draw arc BJ. With center H and radius HF, draw arc FJ. The arcs forming the upper line of the stones are drawn with these centers also.

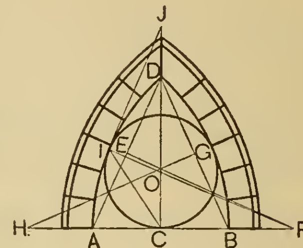
Construct such an arch with any given span and rise. Show that the three arcs form one smooth curve, called a compound curve.



8.* The *Persian* arch with given span and rise is constructed as follows: Let AB be the span, and CD the rise, perpendicular to AB at its middle point. Draw AD and BD. Trisect AD at E and F. Draw $EG \perp AF$, cutting AB at G. Draw $HJ \parallel AB$ through D. Draw GF, cutting HJ at H. Make $DJ = DH$, and $CI = CG$. Draw IJ, cutting BD at K. Now show how to draw the four circular arcs AF, FD, BK, KD. Show that arcs AF and FD have the same tangent line at F, and hence form a smooth curve, called a "reverse curve." Draw a Persian arch with any given span and rise.

Such arches may be drawn by bisecting AD, etc., instead of trisecting it. Draw a Persian arch by making F, the junction point of the arcs AF and FD, the middle point of AD. Explain the construction.

9.* The segmental or *Gothic* arch, with given span AB and rise CD, perpendicular to AB at its middle point, is formed by the arcs of two circles as follows: Draw AD and BD. Draw EF perpendicular bisector of AD, meeting AB at F; and GH perpendicular bisector of BD, meeting AB at H. With center F and radius AF, draw arc AD, and with center H and radius BH, draw arc BD. Construct a Gothic arch with any given span and rise.



Hanstein's "Constructive Drawing" gives the following construction for drawing the inscribed circle of the Gothic arch; *i.e.* a circle touching AB, arc AD, and arc BD each in but one point — part of the inside ornamentation of the arch: Make $CJ = AF$. Make $JI = CF$. Draw IF, cutting CD at O. O is the center of the required circle and OC the radius. Prove it. (Suggestion. — Draw IC. Then $\triangle ICF \cong \triangle ICJ$. $\therefore \angle IJC = \angle IFC$. $\therefore \angle OIJ = \angle OCF =$ rt. $\angle$.)

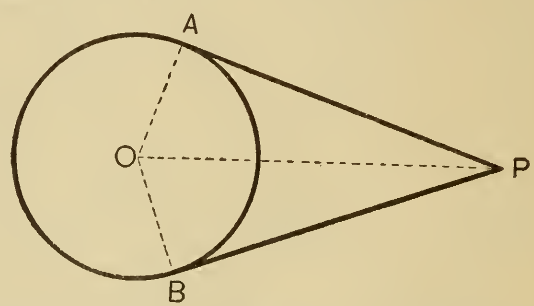
10. Draw two circles having a given straight line as a common tangent, and having a common point of tangency. Having different points of tangency. Are the circles fully determined? Why not?

11. Can two circles always have a common tangent? Explain.

12. Can two circles always have a common chord? Explain.

13. Draw two circles that can have neither a common chord nor a common tangent.

176. Theorem. — *Two tangents to a circle, from a point without the circle, are equal.*

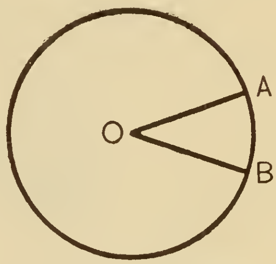


Note. — By the length of a tangent is meant the length of the sect from the point of tangency to the given point, as AP and BP.

Suggestion. How are sects proved equal? Why are the auxiliary lines drawn? Write out the complete proof.

177. Corollary. — *The two tangents from a point without a circle to the circle make equal angles with the straight line joining the point to the center of the circle.*

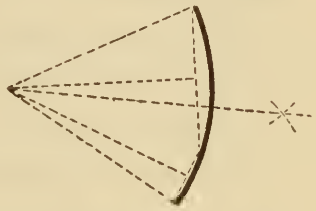
178. Central angles. — Angle BOA is called a **central angle.** What is its vertex? What are its sides? Draw two or more equal circles. Draw any central angle BOA in one. Draw in the same circle and in the equal circles, central angles equal to BOA. Trace the angle BOA and place it upon the other angles. What is true of the arcs? From these tests we may *assume* that:



In the same circle, or in equal circles, equal central angles intercept equal arcs, and conversely.

179. Construction. — The assumption of § 178 suggests a method of bisecting an arc. Study the figure and show how it is done. Bisect a given arc.

Why must you first find the center of the circle of which the arc is a part? How is this done? What is the next step?



NOTE. — For a second method of bisecting an arc, see § 185.

EXERCISES.

1. Divide a semicircle into two equal parts. Into three equal parts.

2. How could you divide a circle into 2, 4, 8, 16, 32, etc., equal parts?

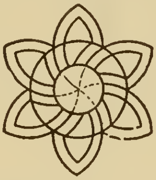
3. Divide a semicircle into 8 equal parts.

4. How could you divide a circle into 3, 6, 12, 24, etc., equal parts?

5. Divide a semicircle into 6 equal parts.

6.* Make an ornamental drawing similar to that in the accompanying figure. The circle is divided into how many equal arcs? How many degrees in each central angle? What kind of triangle is formed by two radii and the chord of the intercepted arc? What other method does this suggest of dividing a circle into six equal arcs?

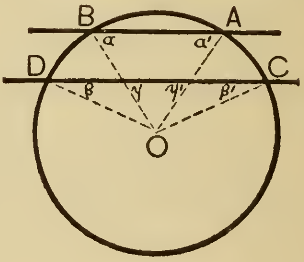
7.* Make ornamental designs similar to the following figures. Describe the construction in each.



180. Theorem. — *Parallel secants of a circle intercept equal arcs.*

Hypothesis. $AB \parallel CD$.

Conclusion. Arc $AC =$ arc BD.



Suggestions. 1. Compare β and β'.

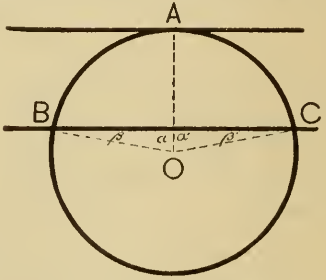
2. Compare α and α'.

3. Compare γ and γ'.

4. Hence, compare $\measuredangle s$ BOD and COA.

5. What is the conclusion?

181. Theorem. — *If a tangent and secant are parallel, they intercept equal arcs on the circle.*

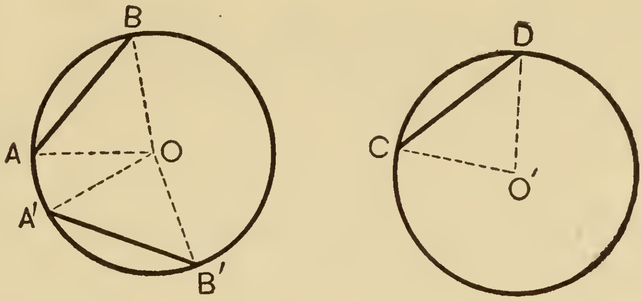


Suggestion. Prove that α and α' are right angles and that $\beta = \beta'$.

182. Theorem. — *If two tangents are parallel, they intercept equal arcs on the circle.*

Suggestion. Draw a secant parallel to one of the tangents, and use § 181.

183. Theorem. — *If two arcs of the same circle or of equal circles are equal, they are subtended by equal chords, and conversely.*



Suggestions. In the first part of the theorem what is given? What is to be proved? How are lines proved equal? Why are the auxiliary lines OA, OB, etc., drawn.

In the second part ("conversely") what is given? What is to be proved? How can arcs be proved equal?

Write out the complete proof.

EXERCISES.

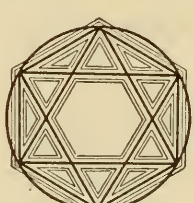
1. The theorem of § 183 suggests a method of drawing an equilateral polygon with its vertices in a circle. Describe the method.

2. Construct in a circle an equilateral triangle.

3. Construct in a circle an equilateral quadrilateral.

4. Construct in a circle a polygon with 6 equal sides. One with 8 equal sides.

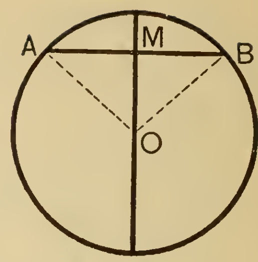
5. The method of Exercise 4 will allow you to construct equilateral polygons with what numbers of sides?



6.* Make an ornamental drawing similar to the one in the adjoining figure. Describe the construction.

Suggestion. — First draw an equilateral polygon with six sides in a circle. Why will joining the alternate vertices form equilateral triangles?

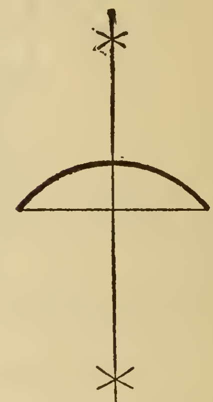
7.* How would you construct a six-pointed star? Construct one.



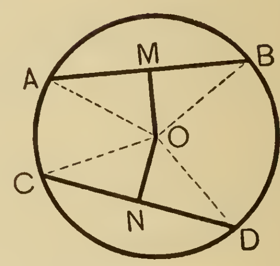
184. Theorem. — *Any diameter perpendicular to a chord bisects the chord and also the subtended arc.*

How will you prove $AM = MB$? What auxiliary lines are necessary? Why? How are arcs proved equal?

Write out complete proof.



185. Construction. — § 168 and § 184 provide a second method of bisecting a given arc, as follows: Draw the chord joining the extremities of the arc, then erect the perpendicular bisector of the chord. This line will bisect the given arc.

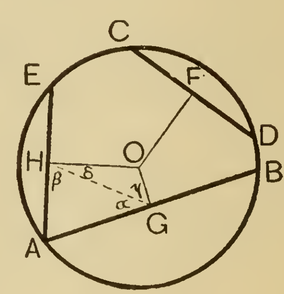


186. Theorem. — *In the same circle, or in equal circles,* (1) *two equal chords are equidistant from the center, and* (2) *conversely, two chords equidistant from the center are equal.*

How are sects proved equal? What auxiliary lines are necessary? Why?

Write out a complete proof for each part of the theorem.

187. Theorem. — *In the same circle, or in equal circles, two unequal chords are unequally distant from the center, and the greater chord is at the less distance.*



Hypothesis. $OG \perp AB$, $OF \perp CD$, $AB > CD$.

Conclusion. $OG < OF$.

Suggestions. 1. Draw AE equal to CD, and $OH \perp AE$. Draw GH.

2. Compare the following in order:

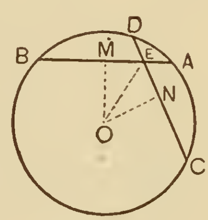
AB and AE, AG and AH, α and β, γ and δ, OG and OH, OH and OF, OG and OF.

188. Theorem. — *In the same circle, or in equal circles, two chords that are unequally distant from the center are unequal, and the chord at the less distance is the greater.*

Suggestion. Use the figure in § 187, reversing the steps in the proof. Or proceed as follows:

1. Either $CD < AB$, $CD = AB$, or $CD > AB$.
2. If $CD = AB$, $OG = OF$, which contradicts the hypothesis. Why?
3. $\therefore CD \neq AB$.
4. If $CD > AB$, $OG > OF$, which contradicts the hypothesis. Why?
5. $\therefore CD \ngtr AB$.
6. $\therefore CD < AB$.

EXERCISES



1. If two equal chords intersect, the sects of one are equal respectively to the corresponding sects of the other.

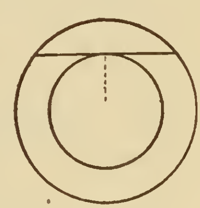
Hypothesis. AB and DC are equal, and intersect at E.

Conclusion. $BE = EC$, and $DE = EA$.

Suggestions. 1. Drop $\perp$s OM and ON from center O, and draw OE.

2. Compare $\triangle$s OEM and NOE.

2. The line joining the center of a circle to the middle of a chord is perpendicular to the chord.



3. Through a given point within a circle draw a chord that shall be bisected at the point.

4. If two circles are concentric (have the same center), a chord of the larger that is a tangent to the smaller is bisected at the point of contact.

5. If two circles are concentric, chords of the larger that are tangent to the smaller are equal.

6. The diameter of a circle bisects all chords parallel to the tangent at the extremity of the diameter.

7. The locus of the middle points of all the equal chords of a circle is another circle having the same center as the first.

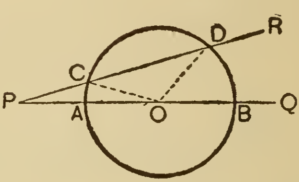
8. Two chords perpendicular to a third chord at its extremities are equal.

9. The diameter is the greatest chord in a circle.

10. The shortest chord that can be drawn through a given point within a circle is perpendicular to the diameter through that point.

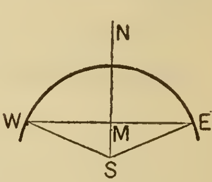
11. If a diameter passes through the middle points of two chords of a circle, the two chords are parallel.

12. Two intersecting chords that make equal angles with the diameter through their point of intersection are equal.



13. PQ is drawn through the center O, intersecting the circle at A and B; and PR is any other secant intersecting the circle at C and D. Prove $PA < PC$, and $PB > PD$.

14. To locate the meridian (a due north and south line) by equal shadows from the sun: A plumb line is supported at S. As the day progresses, the shadow of the plumb line moves from SW to SE. A circle with center S and convenient radius is drawn on the ground. The points W and E where the shadow crosses the circle, the two observations being made at equal intervals before and after noon (say, 10:30 A.M. and 1:30 P.M.), are marked. On June 21 and December 21 the line WE is a due east and west line. Show that if M is the middle point of WE, on either of these dates, and SN drawn through M, then SN is the meridian.



189. Theorem. — *Two circles cannot intersect in more than two points.*

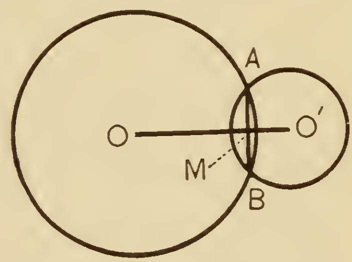
For, if they have more than two points in common, they coincide throughout, since three points not in a straight line determine a circle.

When will they intersect in two points? In one point (two coincident points)? Not at all?

Give the answers in terms of the radii and the distance between the centers.

190. Line of centers. — The straight line joining the centers of two circles is called the **line of centers.**

191. Theorem. — *The line of centers of two intersecting circles is perpendicular to their common chord and bisects it.*

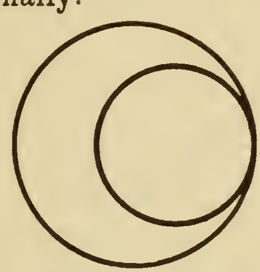


Hypothesis. The common chord AB is cut by the line of centers OO' in M.

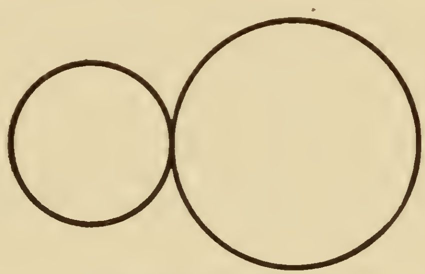
Conclusion. $AM = BM$, and $OO' \perp AB$.

Suggestions. 1. Compare OA and OB, and $O'A$ and $O'B$. 2. See § 113.

192. Tangent circles. — If two circles touch in but one point, they are said to be **tangent** to each other. If one of the two tangent circles lies within the other, they are **tangent internally**, and if each lies without the other they are **tangent externally**.



Tangent Internally.



Tangent Externally.

193. Theorem. — *If two circles are tangent to each other, the point of tangency lies on the line of centers.*

Suggestion. Prove by *indirect method.*

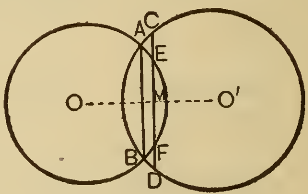
194. Theorem. — *Two circles tangent to each other have a common internal tangent.*

Suggestion. A straight line perpendicular to the line of centers and through the point of tangency is tangent to both circles.

195. Theorem. — *If two circles are tangent to each other, the distance between their centers is equal to the sum or to the difference of their radii.*

EXERCISES

1. If two circles intersect and a straight line be drawn parallel to their common chord and cutting both circles, the intercepted sects between the two circles are equal.



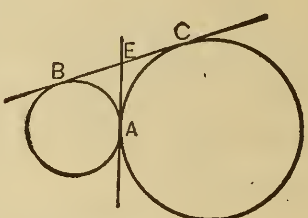
To prove $EC = FD$.

SUGGESTIONS. — 1. Compare CM with MD.

2. Compare EM with MF.

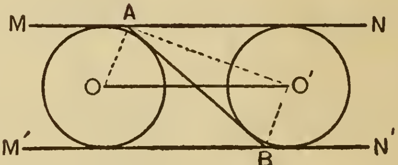
3. From these, compare EC and FD.

2. Two circles are tangent externally at A, and also have a common tangent touching them at B and C, respectively. Prove that the common tangent at A bisects BC.



3. If two circles are tangent externally at A, and have a common tangent touching them at B and C, respectively, a circle with diameter BC will pass through A.

4. If two circles are each tangent to two parallel lines, and also to a transversal of them, the sect of the transversal intercepted by the parallels is equal to the distance between the centers.



SUGGESTIONS. — 1. Compare $\angle MAO$ with $\angle OAB$.

2. Compare $\angle BAO'$ with $\angle O'AN$.

3. Thus compare $\angle OAO'$ with a rt. $\angle$.

4. Also compare $\angle AO'B$ with $\angle OAO'$.

5. Compare $\triangle OAO'$ with $\triangle ABO'$.

6. What is the conclusion?

5. If two equal circles are tangent at A, and any line is drawn through A intersecting the circles again at B and C, respectively, the chords AB and AC are equal.

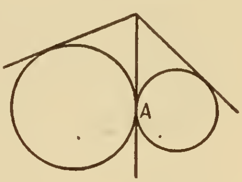
Suggestion. — Draw the common tangent.

6. If two circles meet on their center line, they are tangent.

Suggestion. — Use indirect method.

7. The common tangent of two equal circles that cuts the line of centers between the circles bisects the line of centers.

8. If two equal circles intersect, the common chord bisects the line of centers.



9. If two circles are externally tangent at A, the tangents to them from any point in the common tangent at A are equal.

Does this proof hold when the circles are internally tangent?

10. If two circles are externally tangent, and a line is drawn through the point of contact, terminating in the circles, the diameters drawn through the extremities are parallel.

Suggestion. — Draw the line of centers.

Does the proof hold if the circles are internally tangent?

11. If two cogwheels mesh, the point where they mesh is in a straight line with the center of the wheels.

12. If a triangle ABC is formed by the intersection of three tangents to a circle, two of which, AD and AE, are fixed, while the third, BC, touches the circle at a variable point F on the arc DE, prove that the perimeter of the triangle ABC is constant, and equal to $AD + AE$.

13. Three equal circles are tangent to each other. Prove that the three common internal tangents meet at a point which is equally distant from the three points of contact.

Suggestion. — Let two tangents meet, and join their point of intersection with the third point of contact. Also join this point of intersection with the centers, and draw the radii to the points of contact.

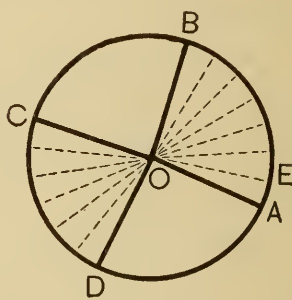
14. The locus of the center of circles tangent to a straight line at a given point is a straight line perpendicular to the line at that point.

15. Show that the locus of the centers of all circles tangent to a given circle at a given point is a straight line determined by the given point and the center of the given circle.

16. What is the locus of the centers of all circles with a given radius r and tangent externally or internally to a given circle? Prove your answer.

17. The locus of the middle points of parallel chords of a circle is the diameter that is perpendicular to any one of them.

196. Theorem. — *In the same circle, or in equal circles, central angles have the same ratio as their intercepted arcs.*



Hypothesis. Angles AOB and COD are central angles, and their intercepted arcs AB and CD.

Conclusion. $\dfrac{\angle AOB}{\angle COD} = \dfrac{\text{arc } AB}{\text{arc } CD}.$

Proof. 1. Let arc AE be a common measure of the arcs.

2. Suppose arc AE contained m times in arc AB and n times in arc CD.

3. Join the points of division to the center O.

4. $\angle AOB$ is divided into m equal angles, and $\angle COD$ into n equal angles. Why?

5. $\therefore \dfrac{\angle AOB}{\angle COD} = \dfrac{m}{n}.$

6. But $\dfrac{\text{arc } AB}{\text{arc } CD} = \dfrac{m}{n}.$

7. $\therefore \dfrac{\angle AOB}{\angle COD} = \dfrac{\text{arc } AB}{\text{arc } CD}.$

NOTE. — Observe that the above proof is not complete, for only the case where the arcs have a common measure is considered. The case where the arcs have not a common measure is too difficult to be treated in this book, although the theorem is equally true in this case.

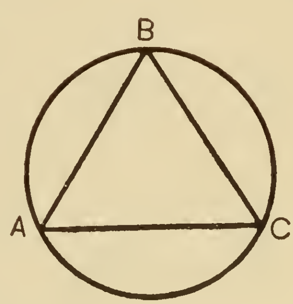
197. Corollary. — *A central angle has the same numerical measure as its intercepted arc.*

For let $\angle COD$ be a unit angle and its intercepted arc CD a unit arc. Then $\frac{\angle AOB}{\angle COD} =$ the measure of $\angle AOB$, and $\frac{\text{arc } AB}{\text{arc } CD} =$ the measure of arc AB.

NOTE. — It is assumed in this corollary, and in all the work that follows, that a unit arc is the arc intercepted by the sides of a central unit angle. If we take for our unit angle $\frac{1}{90}$ of a right angle, or 1°, and draw the radii that divide the whole angle about the center into the 360°, these radii will also divide the circle into 360 equal arcs also called degrees, because equal central angles intercept equal arcs. The number of unit arcs, or degrees, in any given arc is the same as the number of unit angles, or degrees, in the central angle subtended by the arc. Thus, if an angle intercept a sixth of a circle, say, or 60°, the angle is an angle of 60°, or two thirds of a right angle.

198. Supplementary and complementary arcs. — Just as angles have supplements and complements, so have arcs. When two arcs are together equal to a **semicircle** (or 180°), one is said to be the **supplement** of the other; and when two arcs are together equal to a **quadrant** (or 90°), one is said to be the **complement** of the other.

199. Inscribed angles. — The angle formed by two chords meeting on the circle is called an **inscribed angle** of the circle.

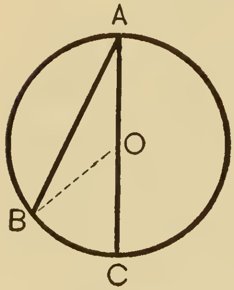


The angle ABC is said also to be inscribed in the **segment** ABC which is formed by the chord AC and the arc ABC. The angle ABC is said to **intercept** the arc AC.

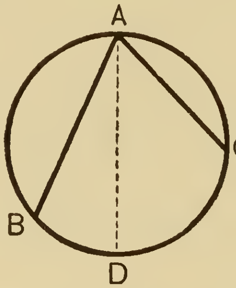
200. Theorem. — *An inscribed angle has the same measure as one half of its intercepted arc.*

There evidently can be three cases, according as the center is (1) on one of the sides, (2) within the angle, or (3) without the angle.

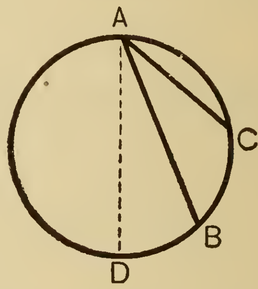
Case I.



Case II.



Case III.



Hypothesis. In Case I, $\angle BAC$ is inscribed in circle with center O, and AC is a diameter.

Conclusion. $\angle BAC$ has the same measure as $\frac{1}{2}$ arc BC.

Suggestions. 1. Compare $\angle BAC$ and $\angle OBA$.

2. Compare $\angle BOC$ and $\angle BAC + \angle OBA$.

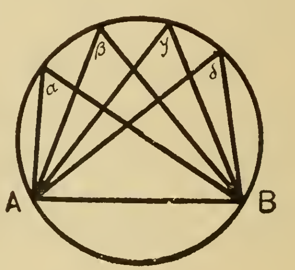
3. Compare the measures of $\angle BOC$ and arc BC.

4. Then compare the measures of $\angle BAC$ and arc BC.

Cases II and III are made to depend upon Case I by drawing diameter AD. Then the given angle is either the sum or the difference of two angles like the angle in Case I. Give the full proof in each case.

201. Corollary 1. — *All angles inscribed in the same segment are equal.*

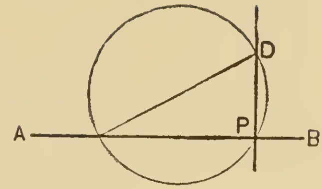
Each of the angles α, β, γ, δ has the same measure as what?



202. Corollary 2. — *An angle inscribed in a semicircle is a right angle.*

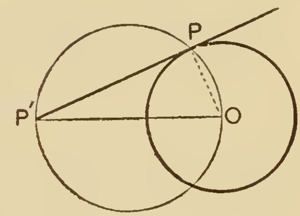
It has the same measure as what?

EXERCISES



1. The corollary of § 202 suggests another method of erecting a perpendicular to a line. Study the figure and show how to erect a perpendicular (1) to AB from point P on the line; (2) to AB from point D without the line.

2. Use the method of Exercise 1 and construct a perpendicular to a line. (Two cases.)

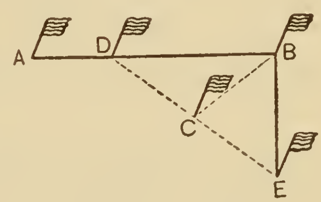


3. Study the figure and show how to construct a tangent from a point to a circle. What two cases?

4. Use the method of Exercise 3 and draw tangents from points to a circle. (Two cases.)

5. When the point is not on the circle, show that two tangents can be drawn from the point to the circle. Draw both.

6. In the figure of Case I, § 200, if arc BC is one fourth of the entire circle, how many degrees in $\angle BAC$?

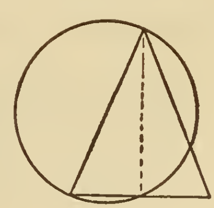


7. Justify the following method by which a surveyor may lay out a line perpendicular to AB at B. Select any convenient point C 50 ft. from B. With one end of the 50-ft. tape at C, swing the other end to D, in line with A and B. With one end of the tape still at C, swing the other end to locate E in line with D and C. Then BE is the required perpendicular.

SUGGESTION. — A circle may be drawn through D, B, and E.

8. An angle inscribed in a segment less than a semicircle is obtuse and an angle inscribed in a segment greater than a semicircle is acute.

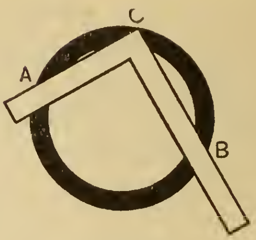
SUGGESTION. — By what are they measured?



9. The bisectors of all angles inscribed in a given segment of a circle meet at one point.

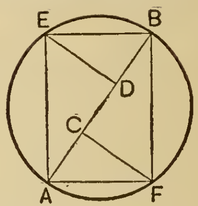
10. If a circle is drawn with one of the equal sides of an isosceles triangle as diameter, it bisects the base.

11. In carpentry, circular pieces of molding for door panels, etc., are sometimes turned out in the form of rings, on a lathe; then these are cut into pieces according to the places in which they are to be used. To cut such a ring into two equal parts, place a carpenter's square upon it with the heel at the edge of the ring, and mark the points A and B where the arms of the square cross the edge of the ring. Show that ACB is one half of the ring.

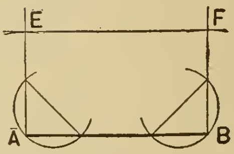


12. Pattern makers and others use the carpenter's square as follows to determine if a half round hole is a true semicircle: The square is placed as in the figure. If the heel of the square touches the bottom of the hole in all positions of the square, while the sides rest against the edges of the hole, the hole is a true semicircle. Justify this test.

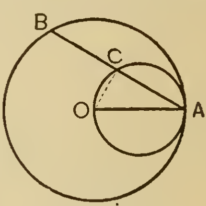
13. The strongest beam that can be cut from a given round log is one in which the breadth is to the depth as 5 is to 7, approximately. It is found graphically as follows: Draw diameter AB and trisect it at C and D. Draw $ED \perp AB$ and $CF \perp AB$. Draw AE, EB, BF, and FA. Show that $AEBF$ is a rectangle.



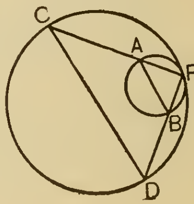
14. In practical work a line EF is drawn parallel to a given line AB sometimes as shown in the figure. Explain the construction, and prove that $EF \parallel AB$.



15. A circle is drawn with the radius of another circle as diameter. Through A, their point of tangency, any chord AB of the larger circle is drawn, intersecting the smaller circle at C. Prove that C bisects AB.

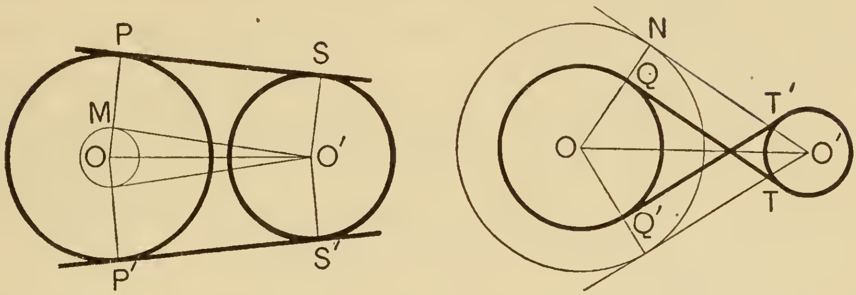


16. Two circles are tangent internally at P. AB is a diameter of the smaller circle. PA and PB intersect the outer circle at C and D, respectively. Prove that CD is a diameter.



203. Construction. — We are now prepared to discuss the following important construction:

To draw the common tangents to two circles.



Given O and O' the centers of the given circles with radii r and r', respectively.

Required to draw the common tangents to the circles.

Suggestion. This may be reduced to Exercise 5, above, by describing concentric to the circle with center O, circles with radii $r - r'$ and $r + r'$.

Construction. — 1. Draw Ⓢ with radii $r - r'$ and $r + r'$ about center O.

2. From center O' draw tangents $O'M$ and $O'N$ to these Ⓢ.

3. Draw OM and ON cutting the circle in P and Q, respectively.

5. Draw $PS \parallel MO'$ and $QT \parallel NO'$. These will be tangent to both circles. Why?

In the same way, draw $P'S'$ and $Q'T'$.

NOTE. — The tangents PS and $P'S'$ are called common **exterior tangents** and QT and $Q'T'$ are called common **interior tangents.**

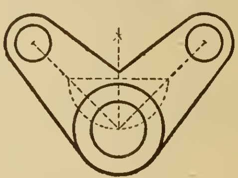
How many common tangents are there, and what kinds, when the circles touch externally? When they intersect? When they become tangent internally? When one circle lies entirely within the other?

Let the student make a figure for each case.

EXERCISES

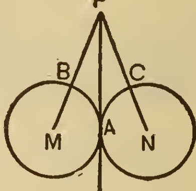
1. A belt runs over two pulleys which are 5 ft. and 2 ft., respectively, in diameter, and the distance between their centers 12 ft. Make drawings to scale for the two cases, when the belt does not cross and when it does. Then, by means of a protractor, measure the arcs of contact of the belt with the pulleys.

2. Construct the pattern of a valve lever with equal arms at right angles, the dimensions being given.

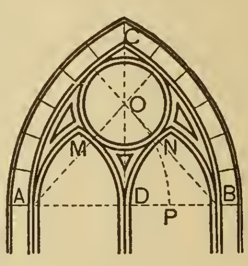


3. The two common internal tangents and the two common external tangents of two circles meet on the line of centers.

4. Two equal circles with centers M and N are tangent at A. The lines PM and PN, joining any point P in the common tangent at A to the centers, cut the circles at B and C, respectively. Prove $PB = PC$.



5. The figure shows a Gothic tracery window. The stone arch ABC consists of two equal circular arcs, and is based upon an equilateral triangle. What are the center and radius for the arc BC? For the arc AC? How are the arches AMD and DNB drawn? Show how to find the center O of the circle tangent to the four arcs MD, ND, AC, and BC. Prove the circle drawn tangent to each of the four arcs. (Suggestion. — P is the middle point of the half span DB. With AP as radius, and A as center, draw arc cutting CD at O. See Exercise 4.)



6. The following figures show designs in window tracery. In Fig. 1, ABC is an equilateral triangle. Show how the figure was constructed. How was the center of the small circle determined? Show that it is tangent to the four arcs.

7. Construct a pattern similar to Fig. 2 with any given span, AB.

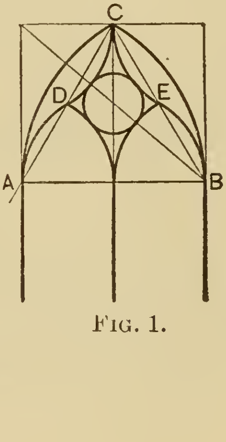


Fig. 1.

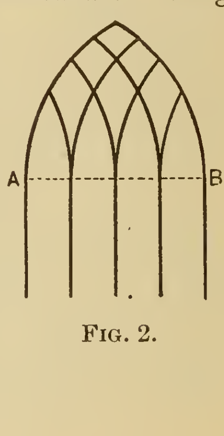
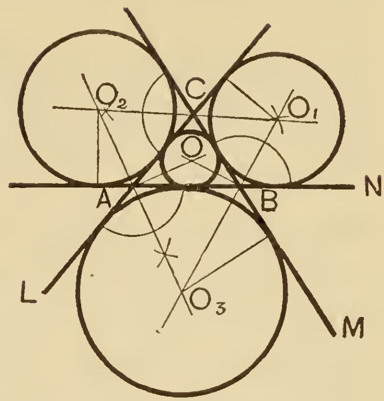


Fig. 2.

204. Construction. — *Draw circles tangent to three given intersecting straight lines.*



Given the straight lines L, M, and N, intersecting in A, B, and C.

Required to construct circles tangent to L, M, and N.

Suggestion. Since the centers must be points equidistant from L, M, and N, the problem reduces to first finding all such points.

Construction. — 1. Bisect the ∡ at A, B, and C, and let the bisectors intersect at O, O_1, O_2, O_3.

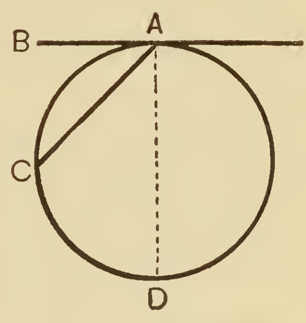
2. Then with these intersections as centers, and with radii equal to the distances from these points to the lines, describe circles. These will be the circles required.

The proof is left to the student.

Can there be any circles if the lines are concurrent? Why?

205. The problem in § 204 might have been stated, *to draw the inscribed and escribed circles of a triangle.* In this case point O is called the **in-center**, and O_1, O_2, and O_3 the **ex-centers**, of the triangle ABC.

206. Theorem. — *An angle formed by a tangent and a chord from the point of tangency has the same measure as half the intercepted arc.*



Hypothesis. BA is a tangent and AC a chord, forming $\angle BAC$.

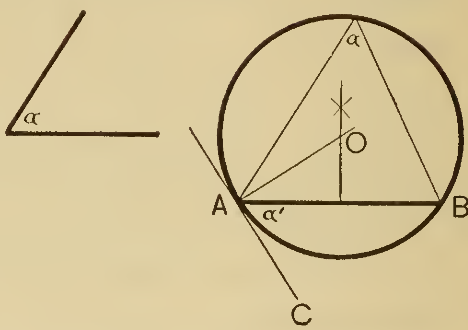
Conclusion. $\angle BAC$ has the same measure as $\frac{1}{2}$ arc AC.

Suggestions. 1. $\angle BAC = \angle BAD - \angle CAD$.

2. $\angle BAD$ has same measure as $\frac{1}{2}$ arc ACD.

207. Construction. — The theorem of § 206 suggests the following problem:

Upon a given sect to describe the segment of a circle that shall contain a given angle.



Given the sect AB and the angle α.

Required to describe the segment of a circle upon AB that shall contain α.

Suggestion. Since an inscribed angle, and an angle made by a chord and a tangent at its extremity, have each the same measure as half the intercepted arc, we have simply to find the center of a circle having AB as a chord and as a tangent at A (or B) a straight line making with AB an angle equal to α.

Construction. — 1. Through A draw AC making with AB angle α' equal to α.

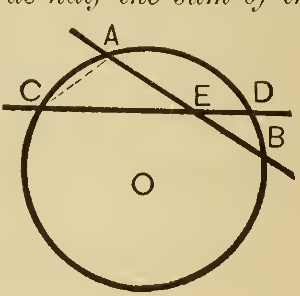
2. Erect the perpendicular bisector of AB.

3. Also erect a perpendicular to AC at A, and let these intersect at O.

4. With O as center and OA as radius describe a circle.

Why will this give the segment required?

208. Theorem. — *The angle formed by two secants intersecting within the circle has the same measure as half the sum of the intercepted arcs.*



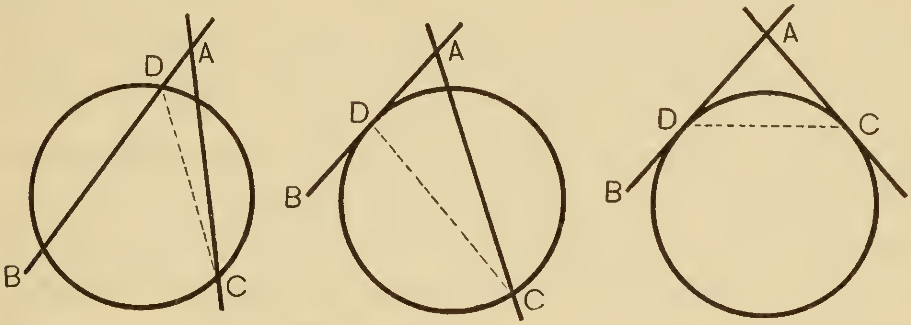
Hypothesis. CD and AB are secants intersecting at E.

Conclusion. The measure of $\angle DEA$ is half that of arc AD + arc CB.

Suggestion. 1. Connect A and C.

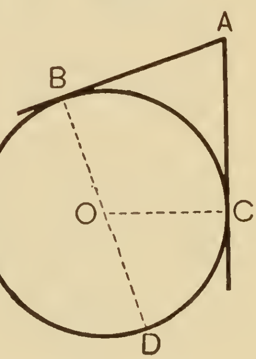
2. Compare $\angle DEA$ with $\angle$s A and C.
3. $\angle$s A and C have the same measures as what?

209. Theorem. — *The angle formed by two secants, or a secant and tangent, or two tangents, intersecting without the circle has the same measure as half the difference between the intercepted arcs.*



In each figure, draw DC, and compare $\angle A$ with $\angle BDC$ and $\angle C$.

210. Corollary. — *The angle formed by two tangents has the same measure as the supplement of the smaller arc.*



Suggestions. 1. Draw diameter BD and radius OC.

2. Compare $\angle A$ with $\angle DOC$ by comparing both with $\angle COB$.

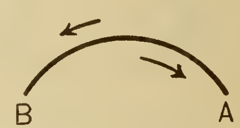
3. Compare arc DC with arc CB.

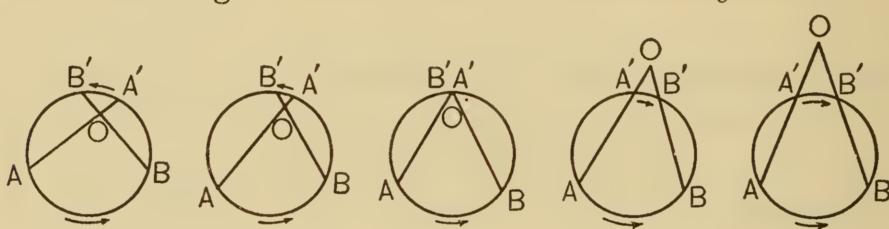
4. What conclusion follows?

211. Positive and negative magnitudes. — It may have been noticed that the theorems in §§ 200, 206, 208, 209, are very closely related. In fact they are seen to be but different special cases of the same general theorem if we take into consideration negative magnitudes.

Since magnitudes are capable of existing in two opposite senses, as *gains* and *losses*, *above* zero and *below* zero on a thermometer, *north* latitude and *south* latitude, etc., it is convenient to consider them as **positive** or **negative**. The student is acquainted with positive and negative numbers in algebra.

The idea of positive and negative magnitudes is carried into other branches of mathematics. In more advanced works we are not only concerned with the length of a line but the direction in which it is drawn; not only the magnitude of an angle but the direction the generating arm moves, etc. Thus if the sect AB is considered as extending from A to B and is *positive*, when extending from B to A it must be *negative*. In the same way if the arc AB, extending from A to B, is *positive*, when extending from B to A it must be *negative*.

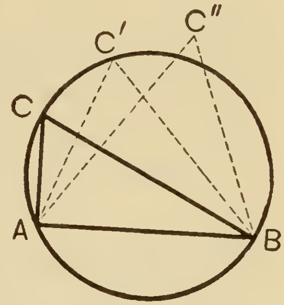




It may be seen in the above figures that while O, the intersection of AA' and BB', is within the circle, the arcs AB and $A'B'$ extend in the same direction. Hence if AB is positive, $A'B'$ is also positive. If A and B remain stationary, arc $A'B'$ becomes smaller as O approaches the circle, and is zero when the two lines intersect on the circle. But as O moves without the circle, $A'B'$ extends in the opposite direction from AB, and hence is negative if AB is positive. In that case the difference between arcs AB and $A'B'$ is their *algebraic sum*. Hence, the four preceding theorems may be considered as different special cases of the one theorem :

The angle formed by two lines has the same measure as half the algebraic sum of the arcs which they intercept on a circle.

212. Theorem. — *The locus of the vertices of all angles which are equal to a given angle and whose sides pass through two given points is the arc of a circle terminating in the two points.*



Hypothesis. A and B are two given points, $\angle ACB$ is equal to a given angle, and arc ACB terminates in A and B.

Conclusion. Arc ACB is the locus of the vertices of all angles equal to $\angle ACB$ and whose sides pass through A and B.

Proof. 1. Let C' be any point on arc ACB.

2. Then $\angle AC'B = \angle ACB$.

3. $\therefore$ all vertices of angles whose sides pass through A and B that are on arc ACB belong to the locus.

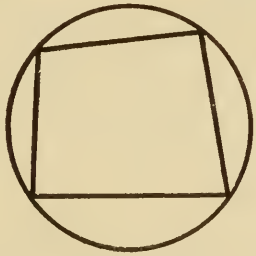
4. Let C'' be any point not on arc ACB.

5. Then $\angle AC''B \neq \angle ACB$, for $\angle ACB$ is measured by $\frac{1}{2}$ arc BA, and $\angle AC''B$ by $\frac{1}{2}$ (arc $BA \pm$ an arc).

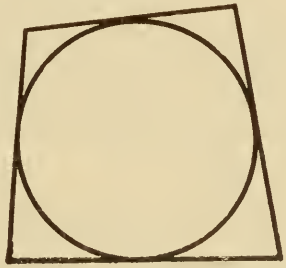
6. $\therefore$ no vertex of an angle whose sides pass through A and B that is not on arc ACB belongs to the locus.

7. $\therefore$ arc ACB is the locus.

213. Inscribed polygons. — If all the vertices of a polygon



An Inscribed Polygon and a Circumscribed Circle.

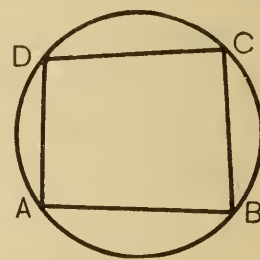


A Circumscribed Polygon and an Inscribed Circle.

lie on a circle, the polygon is said to be **inscribed in the circle,** and the circle is said to be **circumscribed about the polygon.** If a polygon can be inscribed in a circle, it is said to be **inscriptible.**

If the sides of a polygon are all tangent to a circle, the polygon is said to be **circumscribed about the circle,** and the circle is said to be **inscribed in the polygon.** If a polygon can be circumscribed about a circle, it is said to be **circumscriptible.**

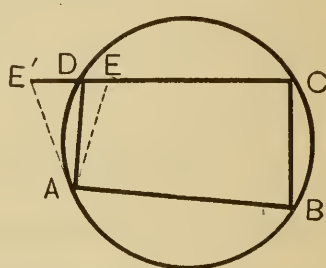
Four or more points through which a circle may be made to pass are said to be **concyclic.**



214. Theorem.—*A quadrilateral inscribed in a circle has its opposite angles supplementary.*

Suggestion. To what is the sum of the arcs which measure $\angle A$ and $\angle C$ equal?

215. Theorem. —*If the opposite angles of any quadrilateral are supplementary, the quadrilateral is inscriptible.*



Hypothesis. In the quadrilateral $ABCD$, $\angle$s B and D are supplementary.

Conclusion. $ABCD$ can be inscribed in a circle.

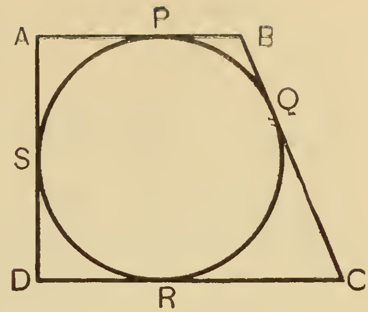
Proof. 1. $\because$ a $\odot$ can be drawn through A, B, and C, suppose it drawn and that it does not pass through D, but cuts CD in E or CD produced in E'.

2. Then $\angle CEA$ or $\angle CE'A = \angle CDA$, each being a supplement of $\angle B$. (Why?)

3. But this is impossible. (§ 64)

4. Therefore the supposition is false, the circle passes through D, and the quadrilateral is inscriptible.

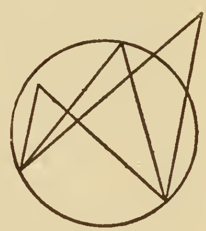
216. Theorem. — *In a circumscribed quadrilateral, the sum of two opposite sides equals the sum of the other two opposite sides.*



Suggestion. Use § 176. Let the student write out the complete proof.

EXERCISES

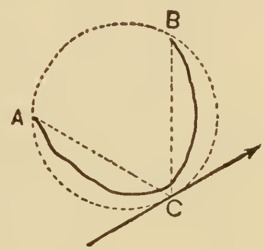
1. An inscribed angle of a circle is less than an angle whose vertex is within the circle and whose sides intercept the same arc, and greater than an angle whose vertex is without the circle and whose sides intercept the same arc.

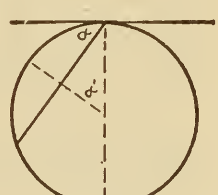


2. If an angle is greater than the inscribed angle of a circle which intercepts the same arc, its vertex is within the circle; if it is equal to the inscribed angle, its vertex is on the circle; and if it is less than the inscribed angle, its vertex is without the circle.

Suggestion. — Use the indirect method in proving each of these parts.

3. A ship is steered past a known region of danger as follows: A chart is made in which a circle is drawn through two points, A and B, which can be seen from the ship, and with sufficient radius that the circle incloses the danger region. The inscribed angle ACB is measured. Observations of A and B from the ship are made from time to time, and the course of the ship directed so that the angle between the directions to A and B never becomes greater than $\angle ACB$. Justify this method.





4. Prove the theorem, § 206, by drawing a radius perpendicular to the chord and a diameter from the point of tangency. ($\angle \alpha = \angle \alpha'$. Why?)

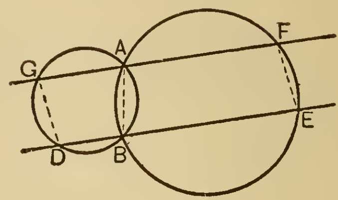
5. Prove the theorem, § 206, by drawing the chord from $C \parallel AB$.

6. Prove the theorem, § 208, by drawing the auxiliary line through one of the extremities of one chord and parallel to the other.

7. Prove the theorem, § 209, by drawing the auxiliary line through D parallel to AC.

8. If a circle can be circumscribed about a parallelogram, the parallelogram must be a rectangle.

9. If two circles intersect, and parallels are drawn through the points of intersection, the sect on one intercepted by the two circles is equal to the sect intercepted on the other.



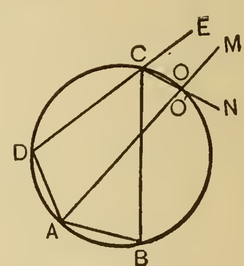
SUGGESTIONS.—1. Draw GD, AB, and EF.

2. Compare $\angle G$ and $\angle ABD$; $\angle ABD$ and $\angle EBA$; $\angle EBA$ and $\angle F$.

3. Compare $\angle$s G and F, of quadrilateral $DEFG$.

4. What kind of figure is $DEFG$?

10. The straight lines that bisect an interior and an opposite exterior angle of an inscribed quadrilateral meet on the circle.



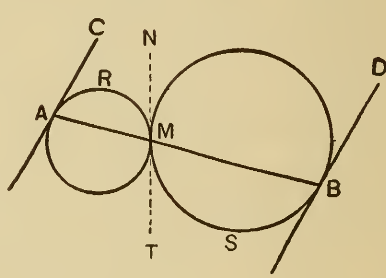
Let AM bisect $\angle BAD$ and CN bisect $\angle BCE$, and let them intersect the circle in O and O' respectively.

To Prove that O and O' coincide.

SUGGESTIONS.—1. Compare $\angle BAD$ and $\angle BCE$.

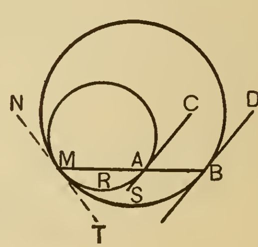
2. Compare arc BO and arc BO'.

11. *If two circles are tangent to each other and a sect be drawn through the point of tangency and terminating in the circles, tangents at the extremities of this sect are parallel.*



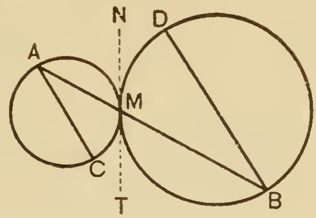
Case I, the circles tangent externally; and Case II, tangent internally.

SUGGESTIONS.—In Case I, compare $\angle NMA$ and $\angle TMB$ and thus compare the measures of arcs MRA and MSB. Then compare $\angle$s MAC and MBD.



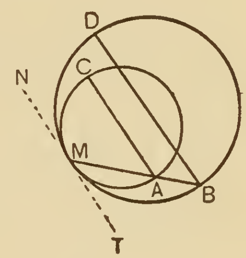
In Case II, prove that arcs MRA and MSB also have the same measure.

12. *If two circles are tangent, and a tangent to one is parallel to a tangent to the other, the straight line joining their points of tangency will pass through the point of tangency of the circles.*



(Use the figures of Ex. 11 and prove by indirect method.)

13. *If two circles are tangent to each other and a sect be drawn through the point of tangency, terminating in the circles, the diameters from the extremities of this sect are parallel.*

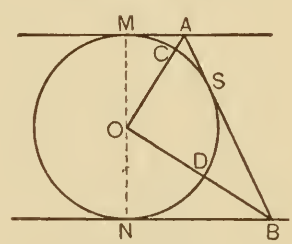


Case I. — Circles tangent externally.
Case II. — Circles tangent internally.

Suggestion. — In both cases prove that arcs MC and MD have the same measure, *i.e.* that they have the same number of degrees.

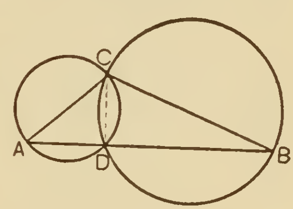
14. *If two circles are tangent externally to each other, the straight line joining the alternate extremities of parallel diameters passes through the point of tangency.*

15. If two parallel tangents intercept a sect on a third tangent, the intercepted sect subtends a right angle at the center of the circle.



Suggestion. — Prove $\angle BOA$ a right angle. How long is the arc CSD?

16. Prove that the sum of all the angles of a triangle is equal to two right angles by circumscribing a circle about the triangle.

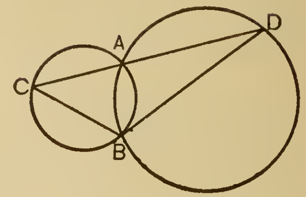


17. In any inscribed trapezoid the non-parallel sides, and also the diagonals, are equal.

18. The circles described on two sides of any triangle as diameters intersect on the third side.

Suggestion. — Draw $CD \perp AB$. Prove that D must lie on each circle, hence, on their intersection.

19. Two circles intersect in A and B. CD is any sect through A terminating in the circles. Prove that for all positions of CD, $\angle DBC$ is constant, *i.e.* has the same size.

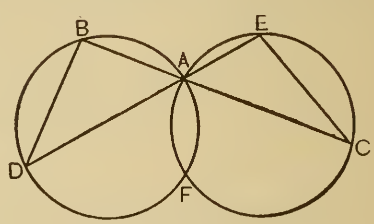


SUGGESTIONS. — 1. $\angle C$, for all positions of CD, has the same measure as half what arc?

2. $\angle D$ has the same measure as half what arc?

3. Why, then, is the size of $\angle DBC$ fixed?

20. Through A, a point of intersection of two equal circles, two sects BC and DE are drawn, terminating in the circles. Prove the chords BD and EC equal.

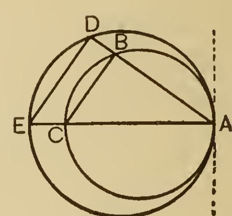


21. In the figure of Ex. 20, if BF and CF, and also DF and EF, are drawn, prove that $BF = CF$ and $DF = EF$.

SUGGESTIONS. — 1. Compare $\angle FBC$ and $\angle BCF$ in $\triangle BFC$.

2. Compare $\angle FDE$ and $\angle DEF$ in $\triangle DEF$.

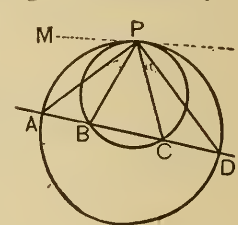
22. Two circles are tangent internally at A. Chords ABD and ACE are drawn cutting the circles at B, D, C and E. Prove $DE \parallel BC$.



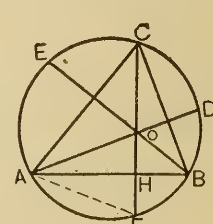
SUGGESTION. — If $DE \parallel BC$, what must be the relation of $\angle AED$ to $\angle ACB$? Then prove $\angle AED = \angle ACB$. How can this be done?

23. Is the above theorem true when the circles are tangent externally? Draw a figure and prove your answer.

24. If two circles are tangent internally at point P, and if any straight line is drawn cutting the circles in A, B, C and D, in order, prove that $\angle APB = \angle CPD$. (Prove that $\angle MPA = \angle D$, and that $\angle PCB = \angle CPD + \angle D = \angle APB + \angle MPA$.)



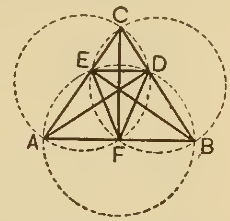
25. If the altitudes of a triangle be produced to meet the circumscribed circle, the sects intercepted by their point of concurrence and the circle are bisected by the sides of the triangle.



SUGGESTIONS. — 1. Since OH is to be proved $= HF$, and since $OF \perp AB$, this suggests that $\triangle FAO$ is isosceles.

2. If $\triangle FAO$ is isosceles, then $\angle BAD = \angle FAH$, which suggests that arcs FB and DB are equal. Prove them so by proving $\angle FCB = \angle BAD$.

26. If the feet of the altitudes of a triangle be joined, they form a triangle whose angles are bisected by the altitudes of the given triangle.

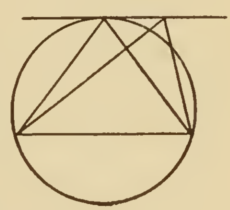


Suggestions. — 1. Describe circles on the sides as diameters.

2. Prove $\angle CAD = \angle EBC$ and hence arc $DC =$ arc EC in degrees, and so on for the other arcs.

27. If through one of the points of intersection of two circles a diameter of each is drawn, the straight line joining the extremities of the diameters passes through the other point of intersection and is parallel to the line of centers. (See Exercise 16.)

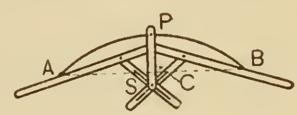
28. If an equilateral polygon is inscribed in a circle, a second circle may be inscribed in the polygon.



29. Of all triangles having the same base and equal altitudes, the isosceles triangle has the greatest vertical angle.

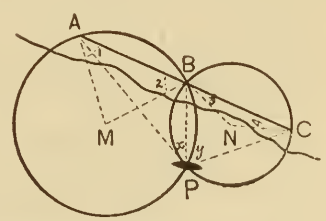
Suggestion. — Circumscribe a circle about the isosceles triangle.

30.* Sometimes in practical work it is necessary to draw the arc of a circle when it is not convenient to find the radius. Show that it may be done as follows: AB is the chord of the required arc and PC the rise. Nails or thumb tacks are placed at A and B. An instrument as shown in the figure is adjusted to the chord and rise and clamped by means of a set screw at S. The instrument carries a pencil at P, and is moved about so that the arms PA and PB constantly touch the nails at A and B. Show that the pencil traces the arc of a circle passing through A and B.



Show how to make a simple instrument for drawing arcs by nailing together three sticks.

31. An important problem involved in marine surveying is to determine the position P of a boat from which soundings are being taken along a coast. The boat moves from position to position, and it is

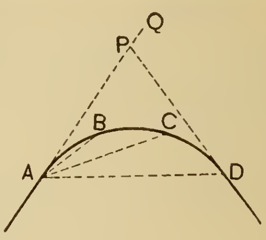


necessary to locate these positions on the chart. Three stations, A, B, and C, are located on the shore. Angles x and y are observed from the ship. A, B, and C are located on the chart. P is located on the chart by the intersection of two circles passing through A, B, and C. The problem is to locate their centers. Suppose $x = 40°$, $y = 70°$.

SUGGESTION. — Find $\measuredangle$ 1, 2, 3, 4.

RAILROAD PROBLEMS

32. In railroad surveying, curves are laid out by turning off equal angles and setting stakes every 100 ft. If the curve begins at A, $\angle PAB$ is turned off from the tangent AP and AB measured 100 ft., then $\angle BAC$ is turned off and BC measured 100 ft., etc., continuing until the curve ends in the tangent DP at D.



Since the curve is to be the arc of a circle, show that $\measuredangle PAB$, BAC, etc., must be made equal, and that each must equal one-half the central angle subtended by a 100-ft. chord.

33. In Exercise 32, the angle QPD formed by the tangents at the end of the railroad curve is called the "intersection angle" of the curve, and angles PAB, BAC, etc., the "deflection angles." Show that the intersection angle equals twice the sum of the deflection angles. Show that the intersection angle equals the central angle subtended by the curve.

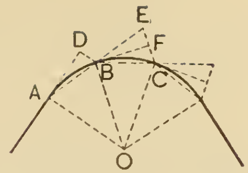
34. To find the length (number of 100-ft. chords) of the railroad track in a curve with a given intersection angle, the curve to be of given "degree." The degree of a curve is determined by a central angle which is subtended by a chord of 100 ft. Thus, a 10-degree curve is one in which a 100-ft. chord subtends a central angle of 10 degrees. If the intersection angle be 34° 20′, and the curve be a 6-degree curve, find the length of the curve.

SUGGESTION. — 34° 20′ = 34.333°. Since each central angle of 6° will subtend a 100-ft. chord, the number of chords will be 34.333° ÷ 6°.

Find the length of a 10-degree curve whose intersection angle is 58° 40′.

35. Surveying railroad curves by "tangent and chord deflections." If AD is tangent to the railroad curve ABC; AB, BC, . . . equal chords; and BD is perpendicular to the tangent AD from the curve; then BD is the "tangent deflection" of the curve. If chord AB is

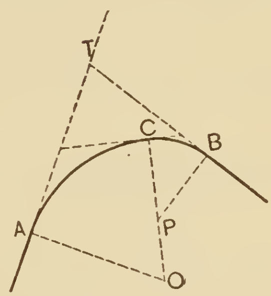
produced to E, making $BE = AB = BC$, the distance CE is the "chord deflection."



Show that (1) the tangent deflection equals one-half the chord deflection; (2) if the length of chord is C and radius R, chord deflection $= \dfrac{C^2}{R}$; (3) tangent deflection $= \dfrac{C^2}{2R}$.

SUGGESTION. — Draw BF tangent at B. Triangles BCE and BCO are similar.

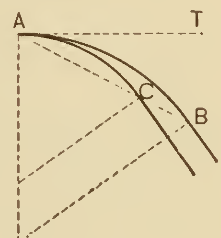
36. In railroad construction, where it is desirable to secure easement in a curve by making the degree of curvature less at the beginning of the curve than farther along, "compound curves" are used. A compound curve is made up of two or more arcs of circles with radii of different lengths, and having common tangents at their junction points. Show that the intersection angle at T equals the sum of the central angles of the arcs AC and CB in the compound curve ACB.



SUGGESTION. — Draw the common tangent at the junction point C.

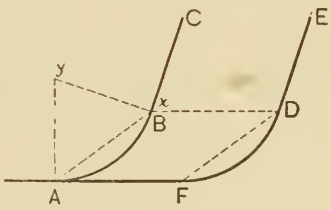
37. Show that the rule given in Ex. 36 holds for a compound curve of three or more arcs.

38. The following principle is used in laying out railroad curves: If the curves of two tracks begin at the same tangent point and end in parallel tracks, (1) their chords coincide in direction, and (2) their chords are proportional to their radii. Demonstrate this.



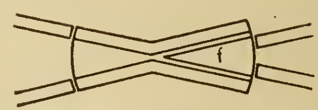
SUGGESTION. — Prove $\angle CAT = \angle BAT$.

39. A railroad curve AB has been laid out, beginning at A and ending in the tangent BC. It is desired to shift the track so that the curve will end in a given tangent DE parallel to BC, which requires shifting the stake A at the beginning of the curve to a new position F. Show that if $\angle x$ is turned off equal to $\angle y$ of the whole curve, and BD prolonged to DE, then the new position F must be such that $AF = BD$.



40. In Exercise 39, show that the construction holds if the curve AB is a compound curve made up of two or more arcs of circles, and $\angle x$ equals the sum of the central angles of the arcs of the curve.

41. In laying out "turnouts," or switches, on a railroad track a "frog" is used at the intersection of the two rails to allow the flanges of the wheels moving on one rail to cross the other. The angle f is called the "angle of the frog", and is the angle at which the rails cross. The angle of the frog that must be selected for any place depends upon the central angles of the two tracks.

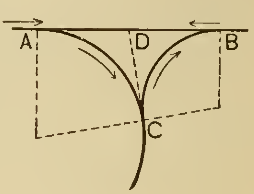


(1) If one track is straight and the other curved, prove that $\angle f$ of frog equals central angle c.

(2) If both tracks curve in the same direction, prove $\angle f = \angle d - \angle c$.

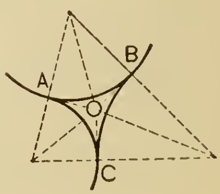
(3) If tracks bend in opposite directions, prove $\angle f = \angle d + \angle c$.

42.* A railroad Y consists of three tracks, AC, CB, and AB, upon which a train is reversed in direction, by moving as shown by the arrows, backing from C to B. In constructing a Y, AB is a straight track, and point A given. The problem is to locate the curves AC and CB with given radii R and r, respectively; *i.e.* to locate stakes C and B.



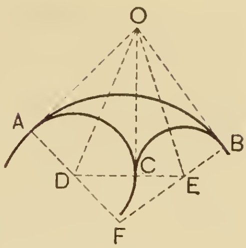
Show that $AD = DC = DB = \sqrt{R \times r}$.

43.* Three railroad curves A, B, and C convex to each other are to form a Y. Show that the lines bisecting the central angles of the curves pass through the intersection of the tangents to the curves, O, and hence that the curves have a common "apex distance" equal to OA.



NOTE. — The determination of the apex distance is a necessary preliminary step in laying out the curves.

44.* A railroad Y with three curves AB, AC, BC, with given radii, is to have curve AB concave to the other two curves. To locate the common "point of intersection" O of tangents to the curves is a preliminary step in laying out the curves. First draw $\triangle DEF$, whose sides are obviously derived from the radii. Prolong side FD and side FE, and bisect the exterior angles, the bisectors meeting at O. Draw bisector of $\angle DFE$ and prove that it goes through O. Draw OA, OB, OC. Prove that these are the tangents from O.



LOCI

45. Find the locus of the centers of circles tangent to two given intersecting straight lines.

46. Find the locus of the centers of circles tangent to a given straight line at a given point in the line.

47. Find the locus of the centers of circles of given radius and tangent to a given straight line; tangent externally to a given circle; internally.

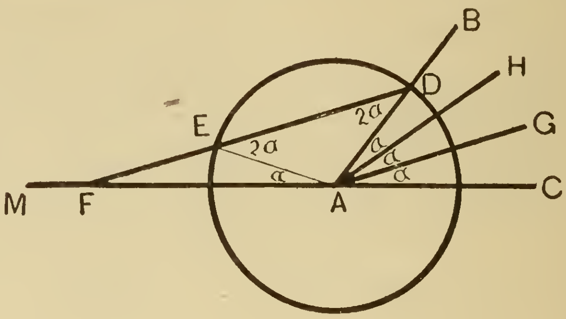
48. Through a point of tangency of two circles a variable sect is drawn, terminated by the circles in A and B. Show that the locus of the middle point of AB is a circle.

49. The locus of the middle points of all chords that subtend a right angle at any fixed point is a circle.

217. Instruments used in problems of construction. — Since the time of Plato, a Greek philosopher and mathematician, born at Athens 429 B.C., the constructions of the science of *elementary geometry* have by common consent been limited to those which may be made by the use of the *straight edge* and *compasses* alone. But, as we have seen, in the practical applications of geometry graduated instruments for measurement are used, such as the ruler graduated according to the French or English units, the protractor, the triangle, etc.

If we are limited to the use of the ungraduated straightedge and compasses alone, there are some constructions that cannot be made.

The following problem, impossible by the use of the compasses and unmarked straightedge alone, is one of the famous problems that mathematicians have worked at unsuccessfully since the time of the early Greeks. The problem is *to divide any angle into three equal parts*. In the following solution, one line must be found by trial, using a graduated ruler not allowed by Plato.



Given the $\angle CAB$.

Required to trisect $\angle CAB$.

Construction. 1. Describe any circle about the vertex A as center, and produce CA through the circle to some point M.

2. Let D be the intersection of the circle with side AB.

3. Now, by trial, by revolving the graduated straightedge about the point D, find a sect from D to a point F on CM such that $FE = EA$.

4. Through A draw $AG \parallel FD$.

5. Bisect $\angle GAB$ by AH, and the given angle is trisected by AG and AH.

Proof. 1. $\angle EFA = \angle \alpha$. (Why?)

2. $\angle AED = \angle \alpha + EFA = 2 \angle \alpha$. (Why?)

3. $\angle ADE = 2 \angle \alpha$. (Why?)

4. $\therefore \angle CAB = \angle EFA + \angle ADE = 3 \angle \alpha$. (Why?)

5. Now $\angle CAG = \angle \alpha$. (Why?)

6. $\therefore \angle GAB = 2 \angle \alpha$. (Why?)

7. $\therefore \angle GAH = \angle HAB = \angle \alpha$, *i.e.* $\angle CAG = \angle GAH = \angle HAB$.

218. The method of constructions. — In the beginning of this book it was shown how to construct certain figures. Later certain exercises were given in which it was proved that these constructions were correct. Many of the theorems studied have suggested methods of constructing certain figures. It is evident that if a certain construction gives a required figure, *the reason for it must depend upon some theorem or theorems.* In order to make a construction the theorem or theorems upon which it depends must be recalled.

It may be necessary in more difficult problems to assume the required construction made (*i.e.* draw the figure free-hand), as in the theorems, and then by analyzing the figure carefully, making any auxiliary lines that are needed, discover the relations existing between the parts of the figure; then, having discovered the key to the construction, reverse the process and construct the figure. This process is called **geometrical analysis**, or the **analytical method**, and is the method usually employed in the solution of problems. The analysis often leads to the simple problem of finding a point when it is known to fulfill two conditions. For example, if it is known that a point lies on each of two intersecting straight lines, it is fully determined, for it must be at their intersection. If it is known that a point lies on a straight line and a circle, or on two intersecting circles, two points satisfy the condition, and there are two solutions. The analysis, the suggestions, and the discussions of the twelve problems that follow this section should give a fairly clear notion of the methods of studying a problem of construction. It should be observed, as the problems are studied, that all problems are reduced to one or both of the fundamental problems:

1. *To draw a straight line between two points*, and

2. *To describe a circle of given radius about a given point as center.*

It should be observed also that either of these depends for its solution upon *finding a point*, and that the required point or points are always found by *the intersection of certain lines*, i.e. *the intersection of known loci.* The loci needed *depend upon some known theorem or theorems.*

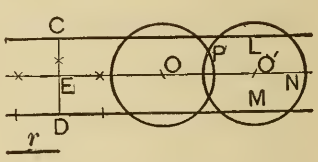
It should be observed also that in the more difficult problems, in order to aid the analysis, the construction is first assumed; that is, the figure is drawn by guess and from its elements known theorems are recalled that make the construction evident.

CONSTRUCTIONS

(To illustrate the method of geometrical analysis.)

1. Construct a circle of given radius that will pass through a given point and cut off equal chords from two given parallel lines.

Let L and M be the given ‖s, P the given point, and r the radius.



Analysis. — 1. The essential thing is to locate the center of the required circle.

2. ∵ equal chords are equally distant from the center, the center must be on the locus of points equidistant from the two ‖s, *i.e.* it must be on a straight line parallel to the given lines and equidistant from them.

3. Also ∵ the circle must pass through point P, the center must be on a circle of radius r described about P as a center.

4. ∴ the required center must be the intersection of the two loci, a straight line, and a circle. Hence, in general, there will be two solutions.

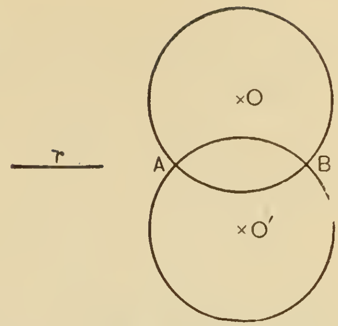
Construction. — (Left to the student.)

Discussion. — If $r = ED$ or $r < ED$, no solution is possible.

Note. — In each problem the pupil should investigate the limits and relations of the data under which the problem shall have a solution. This comes under the head of "Discussion."

2. Draw a circle of given radius that shall pass through two given points.

Let the points be A and B, and the radius r.



Analysis. — 1. ∵ the circle must pass through B, how far is the center from B?

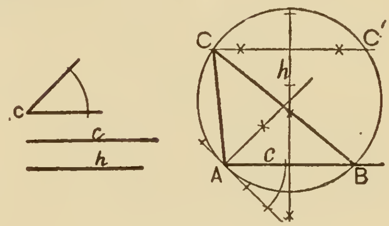
Likewise, how far from A?

2. Upon what two loci, then, does the center lie? How, then, is it found?

Give the steps in the construction.

3. Construct a triangle, having given the base, the angle opposite the base, and the altitude.

Let c be the base, h the altitude and C the angle opposite c.



Analysis. — Suppose that AB is taken equal to c. Now the question arises as to the locus of the vertex of C. ∵ $\angle C$ is constant and AB fixed, the locus of the vertex C under the condition is the arc of a circle through A and B whose segment contains the given angle; and ∵ the altitude $= h$, the vertex must also be in the locus of points equidistant from AB, or a line $\parallel AB$, and a distance h from it.

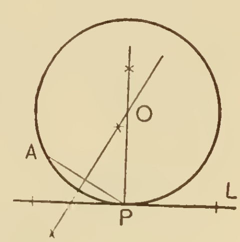
Construction. — (Left to the student.)

Discussion. — (Left to the student. When will there be two △? When one? When is the solution impossible?)

4. Describe a circle that will pass through a given point and be tangent to a given straight line at a given point.

Let L be the given straight line, P the point on the line, and A the other given point.

Analysis. — 1. Since the required circle is to be tangent to L at P, what do you know of the position of the diameter from P with respect to L?



2. Upon what line, then, does the center lie?

3. Since the circle passes through A and P, what do you know of the perpendicular bisector of AP? How, then, can you locate the center?

Give the steps of the construction.

Discussion. — When will the solution be impossible?

5. Describe a circle of given radius which will pass through a given point and be tangent to a given straight line.

Suggestion. — What two loci determine the center?

6. Describe a circle of given radius and tangent to two intersecting straight lines.

Suggestion. — Upon what two loci does the center lie?

7. Describe a circle of given radius tangent to a given circle and to a given straight line.

Analysis. — 1. To be tangent to the given straight line the center must be on the locus of points a constant distance from the given line.

2. To be tangent to the circle the center must be the locus of points a constant distance, which is the sum or the difference of the radii, from the center of the given circle.

Now give the steps in the construction.

Discuss all the possible cases:

(1) When the radius of the given circle is equal to or greater than the radius of the required circle, and the given straight line does not cut the circle.

(2) When the radius of the given circle is less than the radius of the required circle, and the line does not cut the circle.

(3) When the radii are unequal, and the given line cuts the circle.

(4) Radii equal, and the line cuts the circle.

Are there any other possible cases?

8. Describe a circle that shall pass through two given points and have its center on a given straight line.

Suggestion. — How many more loci are needed to determine the center? What is it?

Now give the steps in the construction.

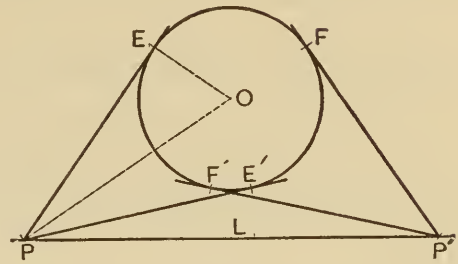
Discuss. — 1. When the given straight line is $\perp$ the line joining the points.

2. When it is not $\perp$ the line joining the points.

3. When the given line is the $\perp$ bisector of the line joining the points.

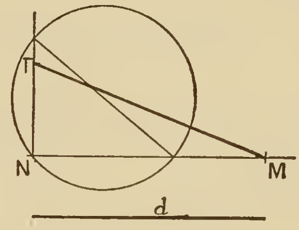
9. Draw a tangent to a circle, such that a sect intercepted between the point of tangency and a given straight line shall have a given length.

Let O be the given circle, L the given straight line, and d the length of the required sect.



Analysis. — 1. The problem is to find a point on L such that tangents from this point to circle O will have the required length d.

2. ∵ a tangent is ⊥ a radius at the point of tangency, the line joining the center of the circle to the point from which the tangent is drawn is the hypotenuse of a rt. △ whose legs are a radius and the length of the tangent.



3. Hence the points from which the tangents are to be drawn are on the circle about O as center and whose radius is the hypotenuse of a rt. △ whose legs are respectively the radius of the circle and d.

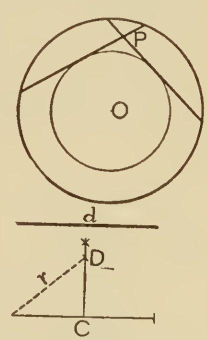
4. Since the circle may cut L in two points, and since there may be two tangents from a point to a circle, four solutions are possible.

Construction. — (Left to the student.)

Discussion. — 1. When will there be but two tangents?

2. When is no solution possible?

10. Through a given point draw a straight line cutting a given circle, so that the part intercepted by the given circle shall have a given length.



Let O be the given circle of radius r, P the given point, and d the length of the required sect.

Analysis. — 1. If the required sect be tangent to a circle whose center is O, it will be ⊥ to its radius at the point of tangency.

2. This radius prolonged being also a radius of the given circle, it will bisect the sect at this point.

3. Hence, since r is known and $\frac{1}{2}d$ may be found, the problem reduces to finding the radius of the circle with center at O and tangent to the required sect.

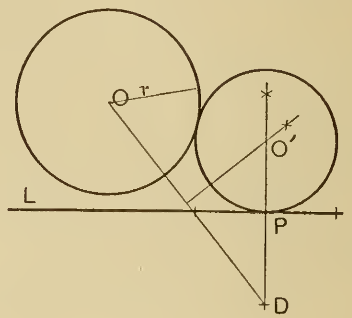
Construction. — (Left to the student.)

Discussion. — 1. When is the problem impossible?

2. What are the limits of the sect d?

3. Discuss the cases when P is within the circle and when without it.

11. Describe a circle tangent to a given circle and to a given straight line at a given point in the line.



Suggestions. — 1. Suppose the problem solved and the figure to be as shown here.

2. Through what point on circle O does *line of centers* OO' pass? Then how long is OO'?

3. Through what point in the required circle will a perpendicular to L at P pass?

4. If O' be extended through P to D, making $PD = r$, and DO drawn, what kind of $\triangle$ is ODO'?

5. How can its vertex, or O', be found when the base OD is known? Give the construction.

Analyze the problem and discuss a solution for a circle tangent internally to the required circle.

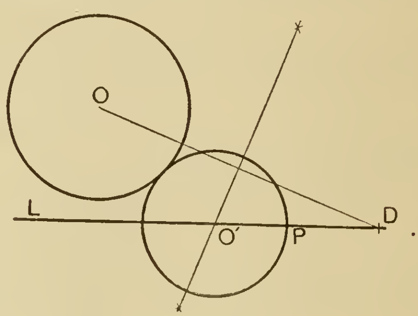
Discussion. — Discuss and draw figures for cases:

(1) When L is without the given circle.

(2) When L is tangent to the given circle.

(3) When L cuts the given circle.

12. Describe a circle through a given point whose center shall lie on a straight line through this point, and which shall be tangent to a given circle.



Let O be the center of the given circle, L the given line, and P the given point.

Suggestions. — 1. Suppose the problem solved and the center of the circle to be O'.

2. What is the length of OO'?

3. Making $PD =$ the radius of O, what kind of triangle is ODO'?

4. How may the vertex, or the center of the required circle, be found?

Give the steps in the construction.

Observe that there was no need of drawing OD, or completing the isosceles △. The construction necessary to find the required center consists merely of drawing what?

Discussion. — Discuss the various possible positions of L and P.

EXERCISES

In the following, let the student discover, by an analysis of the problem, the theorems that may be used, give a construction for each of the possible cases, and a full discussion of each.

1. Construct a triangle having given a side, the angle opposite it, and the altitude to another side.

2. Construct a triangle having given a side, the median to the side, and the altitude to the side. (How many solutions?)

3. Construct a right triangle having given the base and the altitude to the hypotenuse.

4. Construct a right triangle having given the sum of the two legs and an acute angle.

5. Construct a triangle having given two sides and the altitude to one of the given sides.

6. Construct a triangle having given a side and the altitudes to the other two sides.

7. Describe a circle of given radius tangent to two given circles. (How many solutions?)

8. Describe a circle passing through a given point, tangent to a given circle and having its center on a given straight line.

9. Describe a circle passing through two given points and intercepting a given arc on a given circle.

10. Construct a triangle, having given the base, the altitude to the base, and the radius of the circumscribed circle.

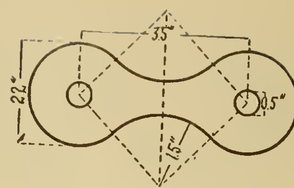
11. Divide a circle into two segments such that the angle inscribed in one shall be double the angle inscribed in the other.

12. Draw a tangent to a circle parallel to a given straight line. Perpendicular to a given straight line.

13. In a given circle draw a chord of given length and parallel to a given straight line. (When impossible?)

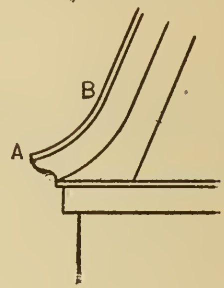
14. Through the intersection of two circles draw a sect of given length terminating in the two circles.

15. Construct the pattern for a link of required dimensions. (The construction is suggested by the diagram.)



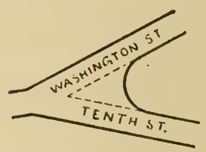
16. In architecture it is required sometimes to draw an easement cornice tangent to the straight or rake cornice at B, and passing through a given point A. Explain the construction, and make such a drawing.

The same construction is used in laying out the easements of stair rails.



17. Draw the connecting curve of given radius r at the intersection of the curbs of two streets.

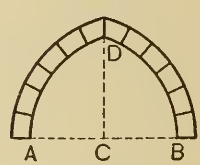
The same construction is encountered in architecture, in drawing the circular line of front of a wedge-shaped building at the intersection of two streets.



18. The construction of a trefoil, formed by the arcs of three equal circles as shown in the figure, is frequently encountered in architectural adornments. Explain the construction, and draw a trefoil with a given radius.

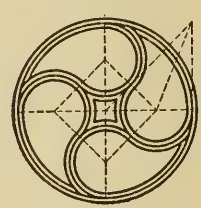
Suggestion. — The figure is based upon an equilateral triangle.

19. Show how to draw a Gothic arch, as shown in the figure, having given the span AB and the altitude CD. Make the construction.

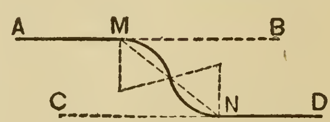


Suggestion. — The arch is based upon an isosceles triangle.

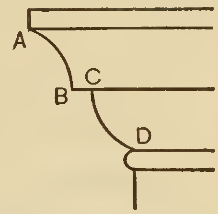
20. Show how to locate the centers of the circular arcs in this rosette. Draw a rosette like this 2 inches in diameter.



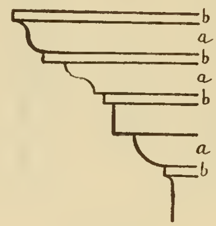
21. AB and CD are parallel. M and N are given points on AB and CD, respectively. Draw a "reverse curve" composed of two equal circular arcs, tangent to AB at M and to CD at N.



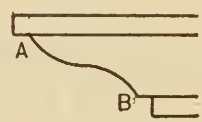
Such curves are constructed in side tracks in railroads, and are used in architecture.



22. Show how to draw the arcs AB and CD of the molding in the figure. Arc AB must pass through the given points A and B and cut the line BC at right angles. Arc CD must pass through the given points C and D and cut the line BC at right angles. Make a complete drawing of such a molding.



23. This figure shows the moldings in the cornice of the Roman Ionic order. The curves are made up of quadrants. Explain the construction, and execute the drawing, making the moldings a each 1 inch wide and the fillets b each $\frac{5}{16}$ inch wide.



24. This is a molding used in Roman architecture, and consists of a reverse curve of two equal arcs joining two given points A and B. Show how it is drawn and make the construction.

25. In this base of a column, Fig. I, the lines AC and BD are given, and it is required to join them by a circular arc with given radius. Show

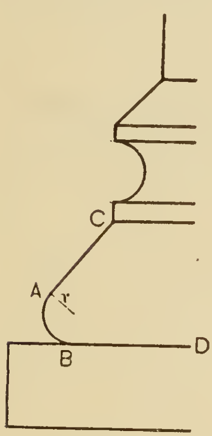


FIG. I

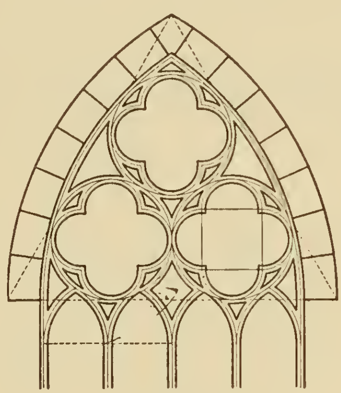


FIG. II

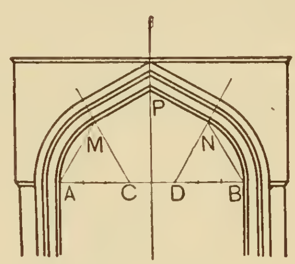


FIG. III

how to make the construction, and make a drawing on a large scale of the whole base of a column like this one.

26. This tracery window, Fig. II, contains three equal tangent circles inscribed in an equilateral triangle. Show how to draw these circles.

27. Construct an arch of the Breccia order as shown in Fig. III. The triangles ACM and DBN are equilateral. MP and PN are tangent to the arcs.

28. The Moorish horseshoe arch, Fig. IV, is constructed as follows: A quadrant OC is drawn and divided into three equal parts, at M and N.

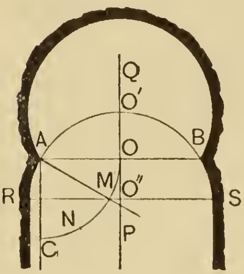


FIG. IV

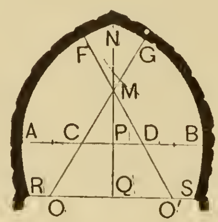


FIG. V

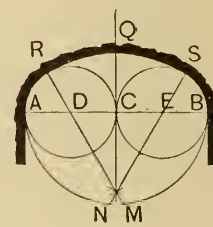


FIG. VI

AM is produced to cut PQ, the perpendicular bisector of span AB, in O''. With O'' as center arc $AO'B$ is drawn, cutting PQ in O'. Now O' is the center of the top arc and O'' is the center of arcs AR and BS. Construct such an arch.

29. From Fig. V discover the construction of the Moorish ogee arch and construct such an arch. ($\triangle CDM$ is equilateral. $MP = PQ$. It is made up of six arcs whose centers are at O', O, A, B, D, and C.)

30. The Tudor arch shown in Fig. VI is made up of four arcs whose centers are at D, E, M, and N. Find the centers and construct such an arch.

CHAPTER IX

METRICAL RELATIONS: CONSTRUCTIONS

NOTE. — Some relations between geometrical magnitudes (sects, angles, arcs, etc.) are represented by corresponding relations between their numerical measures. In the following work, when we speak of the product or ratio of two sects, or other magnitudes, the product or ratio of their numerical measures is meant. Hence, with this understanding, the properties of a numerical proportion proved in algebra hold for a proportion of geometrical magnitudes.

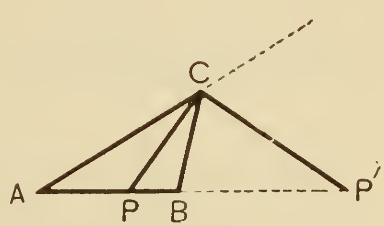
At this point it would be well for the student to review the chapter on proportion in similar triangles.

219. Definitions. — The parts into which a point divides a sect are the portions of the sect, or of the sect produced, between the point and the extremities of the sect.

If the point is on the sect, it is said to divide the sect **internally** into two parts. Thus, the sect AB is divided *internally* by P into parts PA and PB.

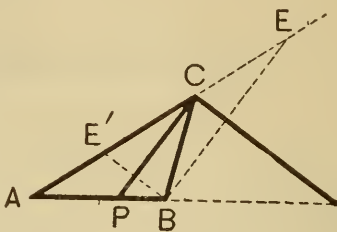
If the point is on the sect produced, it is said to divide the sect **externally** into two parts. Thus, AB is divided *externally* by P' into $P'A$ and $P'B$.

220. Remark. — It is evident that the bisector of an interior angle of a triangle divides the opposite side *internally*, and that in general the bisector of an exterior angle divides the opposite side *externally*.



In case the triangle is isosceles, will the bisector of the exterior angle at the vertex intersect the base prolonged?

221. Theorem. — *In any triangle, the bisector of an interior angle divides the opposite side internally into sects proportional to the adjacent sides, and the bisector of an exterior angle that intersects the opposite side divides it externally into sects proportional to the adjacent sides.*



Hypothesis. In $\triangle ABC$, CP is the bisector of the interior angle at C, and CP' the bisector of the exterior angle at C. P and P', respectively, are their points of intersection with AB.

Conclusion. $\dfrac{PA}{PB}=\dfrac{CA}{CB}$, and $\dfrac{P'A}{P'B}=\dfrac{CA}{CB}$.

Suggestions. 1. Draw $BE \parallel CP$ and $BE' \parallel CP'$.

2. Then $\triangle$ CBE and CBE' are isoscles.

3. See § 124.

222. Theorem. — *If, in any triangle, a straight line from the vertex of an angle divides the opposite side internally into sects proportional to the adjacent sides, it is a bisector of that angle; and if a straight line from a vertex divides the opposite side externally into segments proportional to the adjacent sides, it is a bisector of the exterior angle at that vertex.*

Suggestion. Use the indirect method. Suppose that the lines are not bisectors, assume new lines to be bisectors, form a proportion, and reduce to an absurdity.

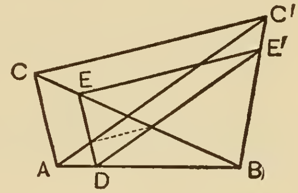
EXERCISES

1. The sides of a triangle are 5, 6, 7. Find the lengths of the sects into which each side is divided by the bisector of the opposite angle.

2. In the same triangle, find the length of the sects into which each side is divided by the bisector of the exterior angle at the opposite vertex.

3. When a sect is divided internally and externally in the same ratio, it is sometimes said to be divided *harmonically*. Show that the bisectors of the interior and exterior angles at any vertex of a triangle divide the opposite side harmonically.

4. From D, any point in the common base AB of two triangles ABC and ABC', DE is drawn parallel to AC, and DE' parallel to AC', intersecting BC in E and BC' in E', respectively. Prove EE' parallel to CC'.



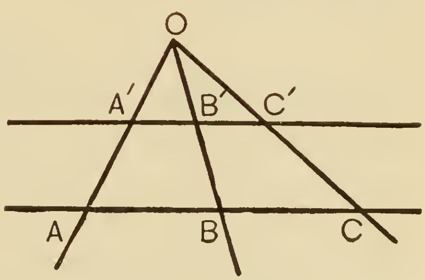
5. If, in two triangles, two sides of one are proportional to two sides of the other, and the angles opposite a pair of homologous sides are equal, the triangles are similar if the angle in each is opposite the longer side; and they may or may not be similar if the angle is opposite the shorter side.

6. In similar triangles, corresponding medians are in the same ratio as any two homologous sides.

7. Are the corresponding bisectors of the angles of similar triangles in the same ratio as any two homologous sides? Prove the answer.

223. A pencil of lines. — If three or more rays are drawn from a point, they are said to form a **pencil of lines.**

224. Theorem. — *If a pencil of lines cut two parallel lines, the corresponding sects intercepted on the parallels are proportional.*



Hypothesis. OA, OB, and OC are a pencil of lines cutting the parallels AC and $A'C'$ in A, B, C and A', B', C', respectively.

Conclusion. $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'}$.

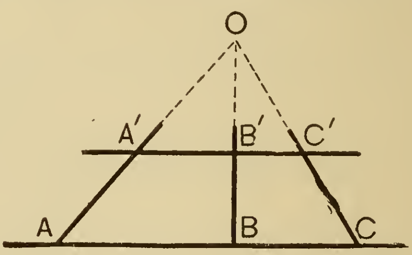
Suggestions. 1. Compare $\triangle ABO$ with $\triangle A'B'O$, and $\triangle BCO$ with $\triangle B'C'O$.

2. Compare $\frac{AB}{A'B'}$ and $\frac{BC}{B'C'}$, each with a ratio of sects common to both pairs of $\triangle$, and hence with each other.

NOTE. — By an extension of this proof, the theorem may be established in the case where more than three rays from O are taken.

EXERCISE. — Draw a figure and discuss the case when three or more lines intersect between two parallels.

225. Theorem. — *If three straight lines intersect two parallels, making the corresponding intercepted sects proportional, the lines are either concurrent or parallel.*



Hypothesis. Parallels AC and $A'C'$ are cut by AA', BB', CC', making $\frac{AB}{A'B'} = \frac{BC}{B'C'}$.

Conclusion. AA', BB', CC' are either concurrent or parallel.

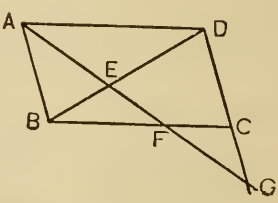
Suggestions. 1. Suppose AA' and BB' not parallel, and produce them to meet at O.

2. Join O to C', and O to C.

3. Then prove that OC contains C'.

4. To do this, compare $\triangle B'C'O$ and $\triangle BCO$, and hence $\angle B'OC'$ and $\angle BOC$.

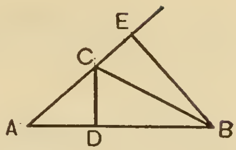
EXERCISES



1. In the parallelogram $ABCD$, AG is drawn cutting the diagonal BD in E, side BC in F, and side DC produced in G. Prove that $\frac{AE}{EF} = \frac{EG}{AE}$.

SUGGESTION.— Compare $\triangle AED$ with $\triangle BEF$, and $\triangle ABE$ with $\triangle EDG$.

2. The altitudes of any triangle are inversely proportional to the corresponding sides.



Hypothesis. — CD and BE are altitudes of $\triangle ABC$.

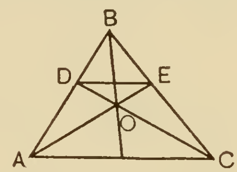
Conclusion. — $\dfrac{CD}{BE} = \dfrac{AC}{AB}$.

Suggestion. — Compare $\triangle$s ACD and ABE.

3. A sect parallel to either base of any trapezoid, terminating in the sides, and drawn through the intersection of the diagonals, is bisected at that point.

4. The diagonals of any trapezoid divide each other into proportional sects.

5. A median to the base of a triangle bisects any sect parallel to the base and terminating in the sides.

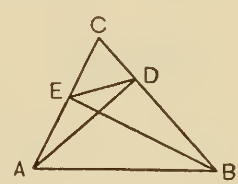


6. In the triangle ABC, DE is parallel to the base AC and intersects AB in D and BC in E. If DC and AE intersect at O, prove BO a median of the triangle. (Use Exercise 3.)

7. If one of the parallel sides of a trapezoid is double the other, their diagonals meet at a point of trisection of each.

8. The straight line joining the middle points of the parallel sides of any trapezoid and the two non-parallel sides produced are concurrent.

9. The straight line joining the feet of the perpendiculars from the extremities of the base of any triangle to the opposite sides make with the sides a second triangle similar to the first.



Hypothesis. — AD and BE are altitudes of any $\triangle ABC$.

Conclusion. — $\triangle EDC \sim \triangle ABC$.

Suggestions. — 1. What angle is common to both $\triangle EDC$ and $\triangle ABC$?

2. Compare $\triangle$s ADC and BEC.

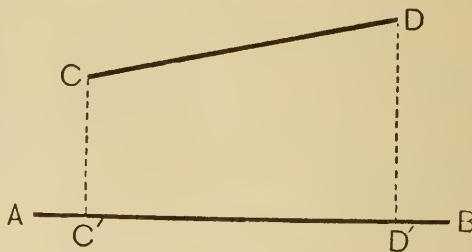
3. Then compare the ratio of the sides including C in each.

10. A diameter AB is produced to a point C. Then CP is drawn perpendicular to AC. PB produced cuts the circle in D. Prove that the triangles ADB and PCB are similar.

Suggestion. — Prove the triangles equiangular.

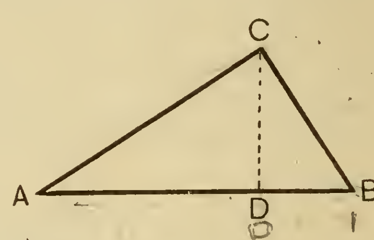
226. Definitions. — The **projection of a point** upon a straight line is the foot of the perpendicular from the point to the line.

The **projection of a sect** upon a straight line is the sect whose extremities are the projections of the extremities of the given sect. Thus, if CC' and DD' are perpendicular to AB, C' is the projection of C, D' of D, and the sect $C'D'$ of sect CD, on AB.



If the relation between three magnitudes, a, b, and x, is expressed by the proportion $\frac{a}{x} = \frac{x}{b}$, x is called the **mean proportional** between a and b, and the proportion is called a **mean proportion.** From this proportion $x^2 = ab$, or $x = \sqrt{ab}$.

227. Theorem. — *In any right triangle, either leg is a mean proportional between the whole hypotenuse and its projection upon the hypotenuse.*



Hypothesis. $\triangle ABC$ is right-angled at C. AD is the projection of AC upon AB, and DB the projection of CB upon AB.

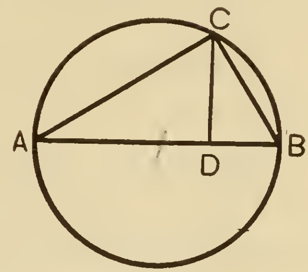
Conclusion. $\frac{AB}{AC} = \frac{AC}{AD}$, and $\frac{AB}{BC} = \frac{BC}{BD}$.

Proof. 1. $\triangle ADC \sim \triangle ABC$ and $\triangle BCD \sim \triangle ABC$.

2. $\therefore \frac{AB}{AC} = \frac{AC}{AD}$ and $\frac{AB}{BC} = \frac{BC}{BD}$.

Or, $\overline{AC}^2 = AB \cdot AD$ and $\overline{BC}^2 = AB \cdot BD$.

228. Corollary. *Any chord is a mean proportional between the diameter and its projection upon the diameter through one of its extremities.*



Suggestions. 1. Let AC be any chord and AB a diameter.

2. What is the measure of $\angle C$? What follows?

229. Theorem. — *In any right triangle, the product of the legs equals the product of the hypotenuse and the altitude to the hypotenuse.*

Suggestion. In § 227, $\triangle ADC \sim \triangle ABC$; and hence $\dfrac{AC}{CD} = \dfrac{AB}{BC}$.

230. Theorem. — *In any right triangle, the altitude to the hypotenuse is the mean proportional between the two segments into which it divides the hypotenuse.*

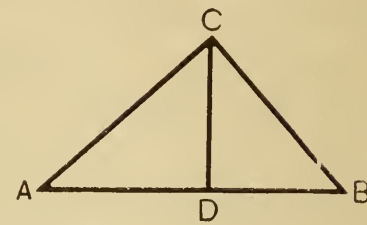
Suggestion. In § 227, $\triangle ADC \sim \triangle CDB$. $\therefore \dfrac{AD}{CD} = \dfrac{CD}{DB}$, or $\overline{CD}^2 = AD \cdot DB$.

231. Corollary. — *The half of a chord perpendicular to a diameter is the mean proportional between the segments into which it divides the diameter.*

232. Theorem. — *The squares of the legs of any right triangle are to each other as their projections upon the hypotenuse.*

Suggestion. In § 227, $\overline{AC}^2 = AB \cdot AD$, and $\overline{BC}^2 = AB \cdot BD$.

233. Theorem. — *In any right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.*



Hypothesis. $\triangle ABC$ is right-angled at C.

Conclusion. $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

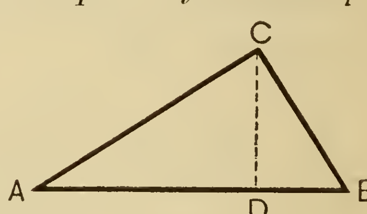
Proof. 1. Drop the perpendicular CD from C to AB.

2. $\overline{AC}^2 = AB \cdot AD$, and $\overline{BC}^2 = AB \cdot DB$.

3. $\therefore \overline{AC}^2 + \overline{BC}^2 = AB(AD + DB) = \overline{AB}^2$.

NOTE. — This theorem is one of the most important and practical ones of geometry. The first proof of it is attributed to Pythagoras, a famous Greek philosopher and mathematician, about 540 B.C. For this reason it is known as the *Pythagorean Theorem.* The truth of the theorem for special cases was known to the Egyptians as far back as 2000 B.C. In laying out their temples, perpendicular lines were found by stretching a rope around three pegs forming a triangle whose sides were proportional to 3, 4, and 5. Carpenters and others often employ the same method yet in making right angles. Two straight lines, one 8 ft. and the other 6 ft., are stretched from a point, and their outer extremities are moved until they are just 10 ft. apart. For another proof of the theorem, see § 280.

234. Theorem. — *In any triangle, the square of a side opposite an acute angle is equal to the sum of the squares of the other two sides, less twice the product of one of these sides by the projection of the other upon it.*



Hypothesis. $\triangle ABC$ is acute-angled at A. The perpendicular from C meets AB at D, between A and B.

Conclusion. $\overline{CB}^2 = \overline{AC}^2 + \overline{AB}^2 - 2\,AB \cdot AD$.

Proof. 1. $\triangle ADC$ and $\triangle CDB$ are rt. $\triangle$s.

2. $\therefore \overline{BC}^2 = \overline{BD}^2 + \overline{CD}^2$.

3. But $\overline{CD}^2 = \overline{AC}^2 - \overline{AD}^2$.

4. $\therefore$ $\overline{BC}^2 = \overline{BD}^2 + \overline{AC}^2 - \overline{AD}^2 = \overline{AC}^2 + (\overline{BD}^2 - \overline{AD}^2)$.

5. But $\overline{BD}^2 - \overline{AD}^2 = (BD + AD)(BD - AD)$

$$= AB(AB - 2\,AD)$$
$$= \overline{AB}^2 - 2\,AB \cdot AD.$$

6. $\therefore$ $\overline{BC}^2 = \overline{AC}^2 + \overline{AB}^2 - 2\,AB \cdot AD$.

Similarly, give the steps of the proof when $\angle B$ is obtuse and D accordingly falls on AB produced.

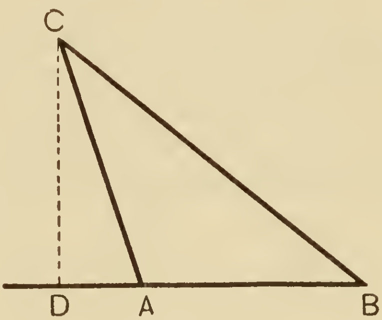
EXERCISES

1. Project AB upon AC and prove the theorem of § 234.

2. Let $\angle B$ be acute and find $\overline{AC}^2$ in terms of $\overline{AB}^2$, $\overline{BC}^2$ and the projection of BC upon AB; also in terms of $\overline{AB}^2$, $\overline{BC}^2$, and the projection of AB upon BC.

Note. — By taking different angles and the projections of different sides, as suggested above, the pupil will become independent of the figure in the text; otherwise he will find this theorem difficult to apply in the theorems and exercises that follow.

235. Theorem. — *In any obtuse triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides plus twice the product of one of these sides by the projection of the other upon it.*



Hypothesis. $\triangle ABC$ is obtuse-angled at A.

Conclusion. $\overline{BC}^2 = \overline{AC}^2 + \overline{AB}^2 + 2\,AB \cdot AD$.

Suggestion. The perpendicular CD will fall without the triangle. With the two rt. $\triangle$ CDA and CDB proceed as in the preceding theorem. Thus, $\overline{BC}^2 = \overline{BD}^2 + \overline{CD}^2 = \overline{AC}^2 + (\overline{BD}^2 - \overline{AD}^2) =$ etc.

EXERCISE

In the figure of § 235 project BC upon AC, and give the corresponding proof.

236. Theorem. — *An angle of a triangle is acute, right, or obtuse, according as the square of the opposite side is less than, equal to, or greater than the sum of the squares of the other two sides.*

Suggestion. Prove by the indirect method. The angle must be either acute, right, or obtuse. Show that if the square of the side opposite is less than the sum of the squares of the other two sides, the angle can be neither right nor obtuse.

EXERCISES

1. The legs of a right triangle are 12 in. and 9 in., respectively. Find the hypotenuse.

2. In triangle ABC, if $AB = 5$, $BC = 6$, and $CA = 8$, find the nature of each of the angles of the triangle.

3. If $AB = 6$, $BC = 8$, $AC = 10$, find the nature of each of the angles.

4. The legs of a right triangle are 8 and 12 respectively. Find the lengths of their projections upon the hypotenuse.

5. A baseball diamond is a square with 90 ft. to the side. Find the distance across from first base to third base.

6. Tools and other pieces of metal are usually made from stock in the form of cylindrical rods of different diameters. A piece of iron is to be milled (cut into different form) so that one end is a square $\frac{5}{8}$ in. on the side. What size (diameter) stock must be selected? See § 233.

7. A derrick is 60 ft. high, and is supported by three steel cables, each reaching from the top of the derrick to a post in the ground 45 ft. from the foot. Allowing 15 ft. for fastenings, how much steel cable is required?

8. Two forces, one of 750 lb. and the other 1000 lb., act upon an object at right angles to each other. To what single force are they equivalent? (See page 60.)

9. One of two forces which act at right angles to each other upon an object is 12 lb. and their resultant is 15 lb. What is the other force?

10. To ream round holes 2 in. in diameter a square reamer is used. Find the dimensions of the reamer.

11. A stair is to be built between two floors whose distance apart is 12 ft. The height of each riser is to be 8 in. and the depth of each tread 10 in. How long must the carriage for the stair be cut?

12. Show that if the sides of a triangle are numerically a, $\frac{1}{4}a^2 - 1$, and $\frac{1}{4}a^2 + 1$, it is a right triangle.

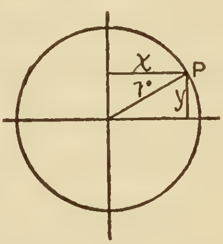
13. Show that $a^2 - b^2$, $a^2 + b^2$, and $2ab$ represent numerically the sides of a right triangle.

14. Pythagoras showed that $2n + 1$, $2n^2 + 2n$, and $2n^2 + 2n + 1$ were the sides numerically of a right triangle. Verify this.

15. This is an ancient Chinese problem: A pool of water was 10 ft. across, and in the middle of it stood a reed which projected one foot above the water. When the wind blew the reed over, the top just reached to the edge of the pool. How deep was the water?

16. Show that the ratio of the diagonal to the side of a square is an incommensurable number, *i.e.* that it cannot be expressed as the ratio of two whole numbers.

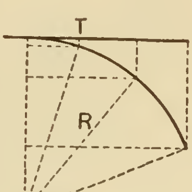
17. Show that if any two perpendicular lines are drawn through the center of a circle, and x and y are the perpendicular distances of any point of the circle from these lines, and r is the radius, then $x^2 + y^2 = r^2$.



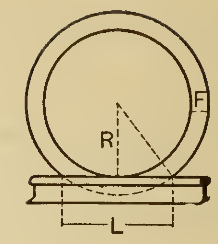
18. A circle may be drawn by locating a large number of its points, and then drawing a smooth curve through these points. By assigning values to y in the equation in Exercise 17, and solving for the corresponding values of x, locate a large number of points of a circle whose radius is 4 in., and by drawing a curve through them construct the circle.

19. The lengths of the radii of two circles are 10 in. and 6 in. respectively, and the distance between their centers 20 in. Find the lengths of their exterior common tangents.

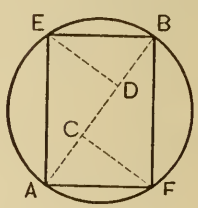
20. Railroad curves are sometimes laid out by measuring offsets from the tangent line at the beginning of the curve. Show that if R is the radius of the curve, and T the distance along the tangent from the beginning of the curve to any offset, then the offset is $R - \sqrt{R^2 - T^2}$.



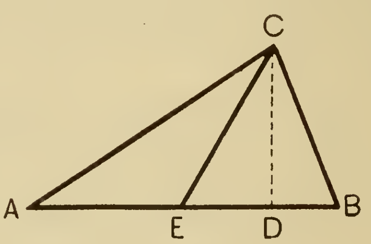
21.* In laying railroad tracks, some roads widen the gauge on curves by a certain amount for each degree of curvature. In figuring the amount to widen the guage, the length L of contact of the flange of the wheel with the rail is used. Show that L is computed by the formula $L = 2\sqrt{(R+F)^2 - R^2}$, where R and F are as shown in the figure. If $F = 1.25$ in., and $R = 2$ ft., compute L.



22. The strongest rectangular beam that can be cut from a given round log is one in which the breadth is to the depth as 5 is to 7, approximately. It is found graphically as follows: Draw diameter AB. Trisect it at C and D. Draw $DE \perp AB$ and $CF \perp AB$. Join E to A, E to B, F to A, and F to B. Show that this construction will give the right dimensions. See § 232.



237. Theorem. — *In any triangle, if a median be drawn to the base,* (1) *the sum of the squares of the other two sides is equal to twice the square of the median plus twice the square of half the base;* (2) *the difference of the squares of the other two sides is equal to twice the product of the base and the projection of the median upon the base.*



Hypothesis. In $\triangle ABC$, CE is the median from C to AB, and ED the projection of CE upon AB. Also $AC > CB$.

Conclusion. $\overline{AC}^2 + \overline{CB}^2 = 2\,\overline{CE}^2 + 2\,\overline{AE}^2$; and $\overline{AC}^2 - \overline{CB}^2 = 2\,AB \cdot ED$.

Proof. 1. $\because AC > CB$, $\angle AEC$ is obtuse.

2. $\therefore \overline{AC}^2 = \overline{CE}^2 + \overline{AE}^2 + 2\,AE \cdot ED$.

3. Similarly, $\overline{CB}^2 = \overline{CE}^2 + \overline{EB}^2 - 2\,EB \cdot ED$.

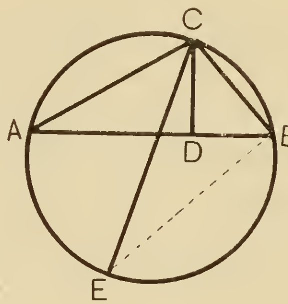
4. $\because AE = EB$, adding, $\overline{AC}^2 + \overline{CB}^2 = 2\,\overline{CE}^2 + 2\,\overline{AE}^2$.

5. Subtracting, $\overline{AC}^2 - \overline{CB}^2 = 4\,AE \cdot ED = 2\,AB \cdot ED$.

Prove the theorem when $AC = CB$. When $AC < CB$.

238. Theorem.—*In any triangle, the product of any two sides is equal to the product of the altitude to the third side by the diameter of the circumscribed circle.*

Hypothesis. In $\triangle ABC$, CD is the altitude from C to AB, and CE is the diameter of the circumscribed circle.



Conclusion. $AC \cdot CB = CE \cdot CD$.

Suggestions. 1. Connect E and B. Compare $\angle s\ A$ and E, and then the rt. $\triangle s\ ACD$ and CEB.

2. Find equal ratios from these triangles that will prove the theorem.

239. Theorem.—*If through any point two secants be drawn to a circle, the product of the distances from the point to the two intersections on one secant is equal to the product of the distances from the point to the two intersections on the other.*

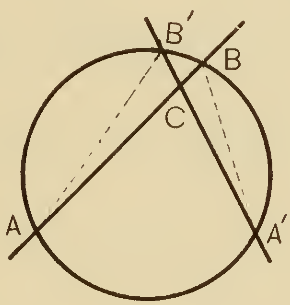


FIG. 1.

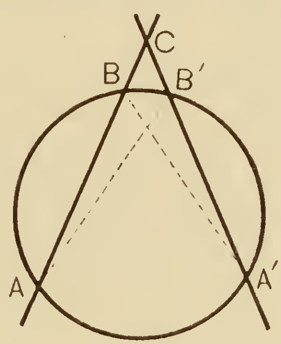


FIG. 2.

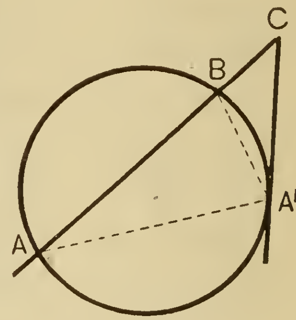
Case I. The point within the circle. (Fig. 1.)

Case II. The point without the circle. (Fig. 2.)

Suggestions. 1. Draw AB' and $A'B$.

2. The theorem can be proved if what relation can be established between the triangles formed?

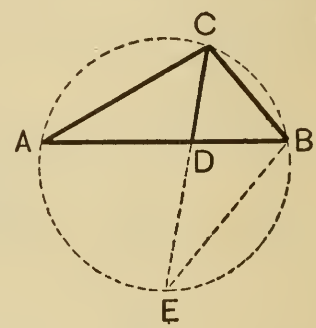
240. Theorem. — *If from a point without a circle a secant and a tangent be drawn, the product of the distances from the point to the two intersections of the secant is equal to the square of the tangent.*



Suggestion. Prove $\triangle AA'C \sim \triangle A'BC$.

Observe that as secant $CA'B'$ in the preceding theorem approaches the position of a tangent, the distances from C to the intersections approach each other. Hence, the tangent CA' may be considered the limiting position of the secant, and this theorem may be proved by making CA' and CB' equal in the preceding.

241. Theorem. — *The square of the bisector of any angle of a triangle is equal to the product of the sides including the angle less the product of the sects of the third side made by the bisector.*



Hypothesis. CD is the bisector of $\angle C$ in $\triangle ABC$.

Conclusion. $\overline{CD}^2 = AC \cdot CB - AD \cdot DB$.

Proof. 1. Circumscribe a circle about ABC; produce CD until it cuts the circle at E; draw EB.

2. $\because \angle A = \angle E$ and $\angle ACD = \angle ECB$, $\triangle ADC \sim \triangle EBC$.

3. $\therefore \dfrac{AC}{CE} = \dfrac{CD}{CB}$.

4. $\therefore AC \cdot CB = CE \cdot CD = (CD + DE)\,CD$
$= \overline{CD}^2 + DE \cdot CD$.

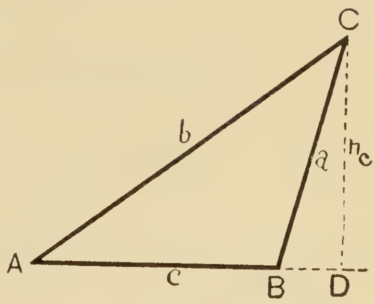
5. But $DE \cdot CD = AD \cdot DB$.

6. $\therefore AC \cdot CB = \overline{CD}^2 + AD \cdot DB$.

7. Or, $\overline{CD}^2 = AC \cdot CB - AD \cdot DB$.

NUMERICAL COMPUTATIONS

242. Computation. — *Compute the altitude of a triangle in terms of its sides.*



At least one base angle must be acute. Let it be $\angle A$.

Let $AC = b$, $AB = c$, $BC = a$, and CD, the altitude from C, $= h_c$.

1. $h_c^2 = b^2 - \overline{AD}^2$.

2. Also $a^2 = b^2 + c^2 - 2c \cdot AD$.

3. $\therefore AD = \dfrac{b^2 + c^2 - a^2}{2c}$.

4. $\therefore h_c^2 = b^2 - \left(\dfrac{b^2 + c^2 - a^2}{2c}\right)^2$

$$= \left(b + \frac{b^2 + c^2 - a^2}{2c}\right)\left(b - \frac{b^2 + c^2 - a^2}{2c}\right)$$

$$= \left(\frac{2bc + b^2 + c^2 - a^2}{2c}\right)\left(\frac{2bc - b^2 - c^2 + a^2}{2c}\right)$$

$$= \frac{(b + c + a)(b + c - a)(a + b - c)(a - b + c)}{4c^2}.$$

5. Now let $a + b + c = 2s$. Then $a + b - c = 2(s - c)$,

$a - b + c = 2(s - b)$, etc.

6. $\therefore h_c^2 = \dfrac{4s}{c^2}(s - a)(s - b)(s - c)$.

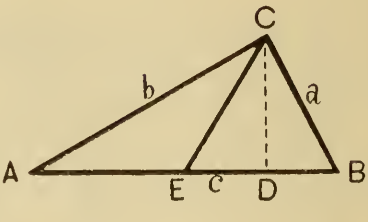
7. $\therefore h_c = \dfrac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}$.

Likewise, $h_b = \dfrac{2}{b}\sqrt{s(s - a)(s - b)(s - c)}$,

and $h_a = \dfrac{2}{a}\sqrt{s(s - a)(s - b)(s - c)}$.

243. Computation. — *Find the length of a median in terms of the sides of a triangle.*

Use the same notation for the sides as in § 242, and let CE, the median from C, equal m_c.



1. $a^2 + b^2 = 2m_c^2 + 2\left(\frac{c}{2}\right)^2.$

2. $\therefore m_c^2 = \frac{a^2 + b^2}{2} - \frac{c^2}{4}.$

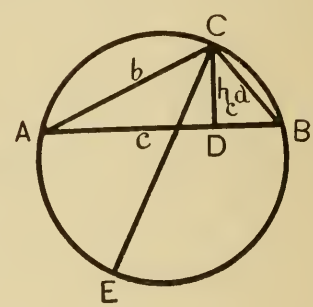
3. $\therefore m_c = \frac{1}{2}\sqrt{2(a^2+b^2) - c^2}.$

Likewise, $m_b = \frac{1}{2}\sqrt{2(a^2 + c^2) - b^2}$;

and $m_a = \frac{1}{2}\sqrt{2(b^2 + c^2) - a^2}.$

244. Computation. — *Find the radius of the circumscribed circle of a triangle in terms of the sides of the triangle.*

Use the previous notation and let $CE = d = 2r$.



1. $ab = d \cdot h_c = 2rh_c.$

2. $\therefore r = \frac{ab}{2h_c}.$

3. But $h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$

4. $\therefore r = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$

245. Computation. — *Find the lengths of the bisectors of the angles of a triangle in terms of the sides.*

Use the previous notation. Let the bisector $CD = t_c$.

1. Then $t_c^2 = ab - AD \cdot DB.$

2. But, $\frac{AD}{b} = \frac{DB}{a} = \frac{AD + DB}{a + b} = \frac{c}{a + b}.$

3. $\therefore AD = \frac{bc}{a + b}$ and $DB = \frac{ac}{a + b}.$

4. $$\therefore\ t_c^2 = ab - \frac{abc^2}{(a+b)^2} = ab\left(1 - \frac{c^2}{(a+b)^2}\right)$$
$$= ab\frac{(a+b+c)(a+b-c)}{(a+b)^2}$$
$$= \frac{ab \cdot 2s \cdot (2s - 2c)}{(a+b)^2} = \frac{4abs(s-c)}{(a+b)^2}.$$

5. $$\therefore\ t_c = \frac{2}{a+b}\sqrt{abs(s-c)}.$$

Likewise, $$t_b = \frac{2}{a+c}\sqrt{acs(s-b)},$$

and $$t_a = \frac{2}{b+c}\sqrt{bcs(s-a)}.$$

EXERCISES: COMPUTATIONS

1. The sides of a triangle are 6, 7, 9. Compute the lengths of

(1) the three altitudes;
(2) the three medians;
(3) the bisectors of the three angles;
(4) the radius of the circumscribed circle.

2. The sides of a triangle are 12, 14, 16. Find the segments of the sides made by the bisectors of the angles.

3. If chord AB of a circle is 8 in. long and 6 in. from the center of the circle, find the radius of the circle.

4. The radius of a circle is 6 in. Through a point 4 in. from the center a diameter is drawn. Find the length of a chord perpendicular to the diameter at this point. Also find the distances from an extremity of the chord to the extremities of the diameter.

5. Through a point a tangent and a secant are drawn to a circle. The secant passes through the center. If the tangent is 10 in. long and the short segment of the secant 4 in., find the radius of the circle.

6. If the radius of a circle is 12 in. and the length of a tangent from a point 15 in., find the lengths of the two segments of the secant drawn from the point through the center.

7. In Problem 6 find the lengths of the segments of a secant whose short segment is 10 in.

8. The sides of an isosceles triangle are equal to a, the base equal to b. Find the altitude to the base.

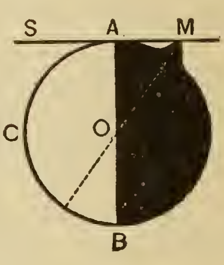
9. If the sides of an equilateral triangle are a, find the altitude in terms of a.

10. The cross section of the train shed of a railway station is to have the form of a pointed arch, made of two circular arcs the centers of which are on the ground. The radius of each arc equals the width of the shed, 210 ft. How long must the supporting posts be made which are to reach from the ground to the dome of the roof?

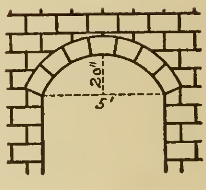
11. If the altitude of an equilateral triangle is h, find the length of the side in terms of h.

12. If r is the radius of a circle, find the length of a chord whose distance from the center is $\frac{1}{2}r$.

13. Galileo (1564–1642) measured the heights of the mountains on the moon, some of which are as much as 7 mi. high, as follows: ACB was the illuminated half of the moon just as the peak of the mountain M caught the beam SM of the rising or setting sun. He measured the distance AM from the half-moon's straight edge AB to the mountain peak M. Then by using the known diameter of the moon, show how he was able to compute the height of the mountain.

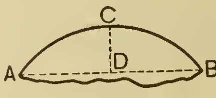


14. A window 5 ft. wide is to be surmounted by a circular stone arch with a rise of 20 in. Find the radius of the circle at which to cut the stone for the arch.



15. In a bridge, a circular arch 18 ft. high is to span a stream 72 ft. wide. What is the radius of the circle at which the stones of this arch must be cut?

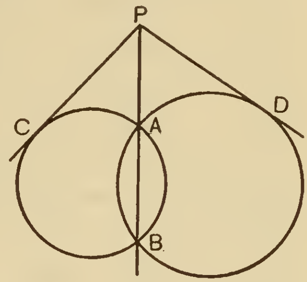
16. This is a piece of a broken wheel. $AB = 16$ in. and $CD = 4$ in. Find the diameter of the wheel. See § 231 or § 239.



18. The diameter of the earth may be computed as follows: Three stakes are set in a canal two miles long, one at each end and one in

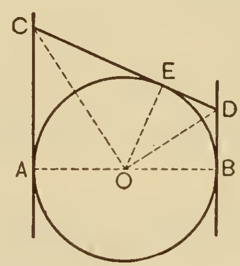
the middle, and all project the same distance above the water. By use of a leveling instrument the middle stake is found to be 8 inches higher than the others. From these facts, compute the diameter of the earth.

EXERCISES: THEOREMS



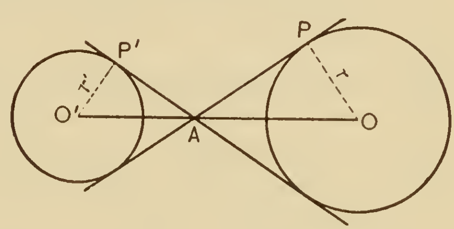
1. Tangents to two intersecting circles from any point on the common chord produced are equal.

2. If the common chord of two intersecting circles be produced to intersect their common tangent, it will bisect the tangent.



3. AC and BD are parallel tangents to circle O. CD is any third tangent intersecting AC in C and BD in D. Prove that $CA \cdot BD$ is constant for all positions of tangent CD.

SUGGESTION. — $\triangle AOC \sim \triangle BOD$.

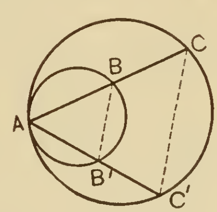


4. The point of intersection of two interior common tangents divides the line joining the centers into sects proportional to the two radii.

(To prove $\dfrac{O'A}{OA} = \dfrac{r'}{r}$.

Prove $\triangle O'P'A \sim \triangle OPA$.)

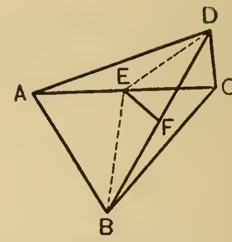
5. The point of intersection of the external common tangents divides the line joining the centers externally into sects proportional to the radii.



6. If two circles are tangent internally, corresponding sects formed by the circles cutting any two lines drawn through the point of tangency are proportional.

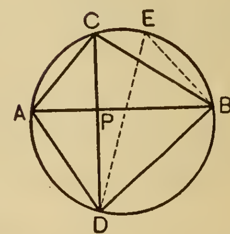
SUGGESTION. — $\triangle ABB' \sim \triangle ACC'$.

7. Discuss Exercise 6, if the circles are tangent externally.



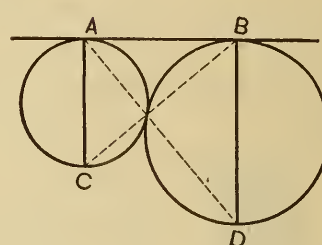
8. In any quadrilateral the sum of the squares of the four sides is equal to the sum of the squares of the diagonals plus four times the square of the sect joining the middle points of the diagonals.

SUGGESTION. — Use § 237.



9. If through any point within a circle two perpendicular chords be drawn, the sum of the squares of the opposite sides of the quadrilateral formed by joining the extremities of the chords is constant for all positions of the point.

SUGGESTION. — Draw $BE = AC$, and prove $\angle EBD$ a rt. $\angle$, and hence ED a diameter.

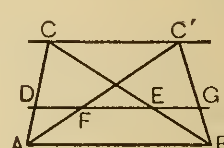


10. The common external tangent to two circles which are tangent externally is the mean proportional between their two diameters.

SUGGESTION. — $\triangle ADB \sim \triangle ABC$.

11. The diagonals of a trapezoid intersect on the line joining the middle points of the parallel sides.

12. If two triangles ABC and ABC' have a common base AB and their vertices C and C' are in a straight line parallel to the base, prove that if any straight line parallel to the base cuts AC in D, BC in E, AC' in F, and BC' in G, then $DE = FG$.



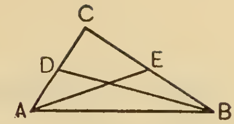
SUGGESTION. — Prove $\frac{DE}{AB} = \frac{FG}{AB}$.

13. If through any point within a circle two perpendicular chords are drawn, the sum of the squares of the four segments is equal to the square of the diameter. (See Exercise 9.)

14. The square of the distance between the centers of two circles cutting each other at right angles is equal to the sum of the squares of the radii.

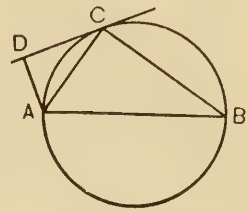
NOTE. — Two circles cut each other at **right angles** when the tangents at the intersection form a right angle.

15. If medians be drawn from the extremities of the hypotenuse of any right triangle, four times the sum of the squares of the medians is equal to five times the square of the hypotenuse.

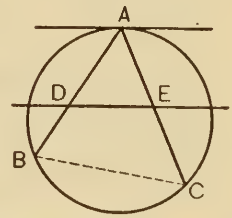


To prove $4(\overline{AE}^2 + \overline{BD}^2) = 5\,\overline{AB}^2$.

16. Triangle ABC is right-angled at C, and AD is the perpendicular from A to the tangent of the circumscribed circle at C. Prove that AC is the mean proportional to AD and AB.



17. If through any point on a circle two chords are drawn, the segments of the chords intercepted between a tangent at the point and any straight line parallel to it are reciprocally proportional to the chords.



To prove $\dfrac{AD}{AE} = \dfrac{AC}{AB}$.

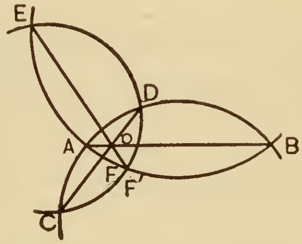
18. In any parallelogram the sum of the squares of the four sides is equal to the sum of the squares of the diagonals. (See Exercise 8, or § 237.)

19. By means of § 233 and § 237, (1), show that in a rigth triangle the median to the hypotenuse is equal to one half of the hypotenuse.

20. Show that the corollary in § 231 follows from § 239.

21. If three circles intersect one another, the three common chords all pass through the same point.

SUGGESTION. — Let the chords AB and CD meet at O. Draw EO and produce it. Suppose that EO produced meets arc EAB again at F and arc EDC at F'. Prove $OF = OF'$, and hence that F and F' coincide.



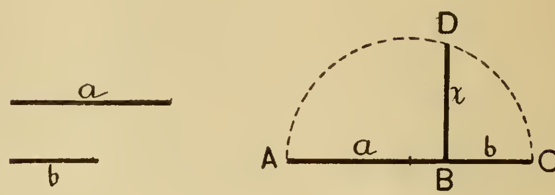
22. The line of centers of two circles meets the circles in A and B, and C and D, respectively, and their common external tangent at E. Prove $AE \times DE = BE \times CE$.

23. If the line of centers of two circles meets the common external tangent at E, and any secant drawn from E intersects the circles at A and B, and C and D, respectively, show that $AE \times DE = BE \times CE$.

CONSTRUCTIONS

246. Construction. — *Find a mean proportional between two given sects.*

First Method.

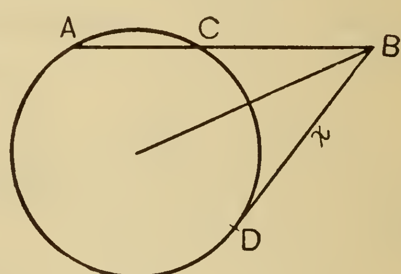


Given the sects a and b.

Required to construct a mean proportional between a and b.

Analysis. Cor. § 231 suggests the construction. Give it.

Second Method.



Analysis. § 240 also suggests a construction. Give it.

247. Extreme and mean ratio. — A sect is said to be divided into **extreme and mean ratio** by a point when one part is the mean proportional between the whole sect and the other part. Thus, sect AB is divided by P *internally* into extreme

and mean ratio if $\dfrac{AB}{AP} = \dfrac{AP}{PB}$, and by P' *externally* into extreme and mean ratio if $\dfrac{AB}{AP'} = \dfrac{AP'}{P'B}$.

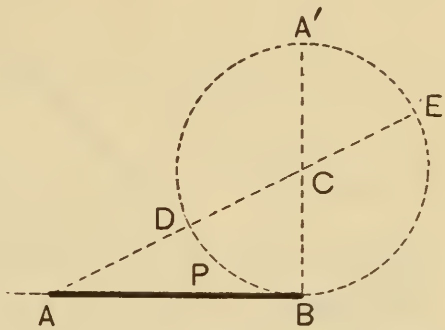
Note. — The problem to divide a sect into extreme and mean ratio is often called the *Problem of the Golden Section.*

248. Construction. — *Divide a given sect into extreme and mean ratio.*

Given sect AB.

Case I.

Required to divide AB *internally* into extreme and mean ratio; that is, to find a point P on AB such that $\frac{AB}{AP}=\frac{AP}{PB}$.



Analysis. 1. The proportion does not suggest directly a theorem upon which the construction depends. Hence we transform the proportion into equivalent proportions until one is found that does suggest the needed proportion.

2. The given proportion is equivalent to $\frac{AB}{AP}=\frac{AP+AB}{AB}$, which is true of a tangent AB and a secant from A whose segments are AP and $AP+AB$. Hence, the following construction:

Construction. 1. Draw a circle tangent to AB at B and with a diameter equal to AB.

2. From A draw a secant through the center of the circle, and let it cut the circle in D and E.

3. Lay off on AB, $AP = AD$. Then P will be the point required.

Proof. (Left to the student.)

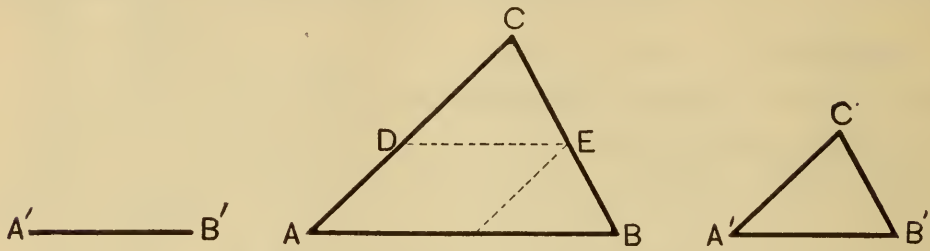
Case II.

Required to divide AB *externally* into extreme and mean ratio. That is, to find a point P' on BA produced through A such that $\frac{AB}{AP'}=\frac{AP'}{P'B}$.

Construction. Lay off on BA produced, $AP' = AE$.

(Let the student produce the line and prove.)

249. Construction. — *Upon a given sect construct a triangle similar to a given triangle and such that the given sect shall correspond to a given side of the given triangle.*



Given $\triangle ABC$ and sect $A'B'$.

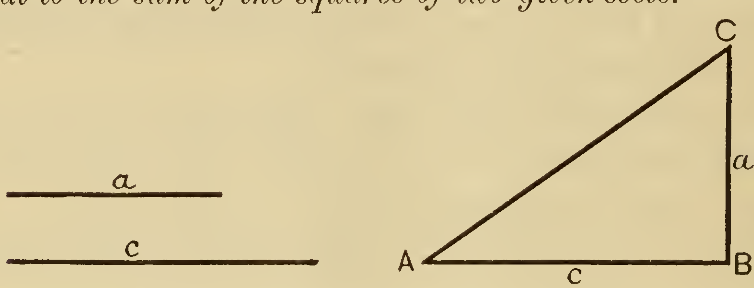
Required to construct upon $A'B'$, as a side corresponding to AB, a triangle similar to $\triangle ABC$.

Suggestion. $\angle A' = \angle A$, and $\angle B' = \angle B$.

250. Construction. — *Upon a given sect construct a polygon similar to a given polygon.*

Suggestion. Use § 135 and § 249.

251. Construction. — *Construct a sect such that its square will be equal to the sum of the squares of two given sects.*



Given sects a and c.

Required to construct a sect whose square is equal to the sum of the squares of a and c.

Suggestion. The only theorem in which the square of one sect has been proved equal to the sum of the squares of two sects is in § 233.

252. Construction. — *Construct a sect such that its square will be equal to the difference of the squares of two given sects.*

(Let the preceding problem suggest a construction.)

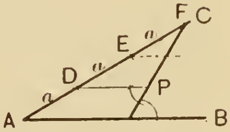
EXERCISES

1. Trisect a given sect by theorem, § 221.

2. Construct a triangle having given the vertical angle, the base, and the ratio of the other two sides.

3. Show how to cut off $\frac{2}{5}$ of a sect.

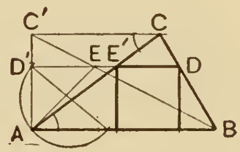
4. Through a point between two intersecting straight lines, draw a straight line whose intercepted sect shall be divided by the point in the ratio 1 : 2.



SUGGESTION. — Draw through P, $PD \parallel AB$.

5. Inscribe a square in a given triangle.

ANALYSIS.—1. In a right triangle if two sides of the square coincide with the two legs, the vertex opposite the right angle of the square will be at the intersection of the bisector of the right angle and the hypotenuse. (Why?)



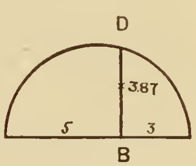
2. Also if two $\triangle$ ABC and ABC' have a common base AB, and their vertices C and C' in a straight line parallel to AB, and any straight line DD' cuts the sides in D', E, E', D, then $D'E = DE'$.

Then how shall DD' be drawn so that DE' shall be one side of the required square?

If one angle of the given triangle is obtuse, what side will you select for the common base of the triangles?

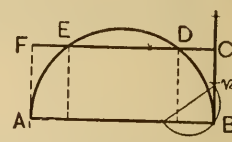
6. Find by construction the square root of a number, as 15.

Since $15 = 5 \times 3$, $\sqrt{15}$ is the mean proportional to 5 and 3. Hence construct BD the mean proportional to 5 and 3. By measuring BD with the diagonal scale, $BD = 3.87$, approximately, the square root.



7. By the method of Problem 6, find the square root of 12; 20; 84; 7; 11; 19. (Consider $7 = 7 \times 1$; etc.)

8. The quadratic equation $x^2 - 5x + 4 = 0$ may be solved geometrically as follows: It may be written in the form $x(5 - x) = 4$. Draw AB equal to 5 units. Erect a semicircle upon AB as diameter. Draw $BC \perp AB$, and mark off $BC = \sqrt{4}$. Through C draw $CF \parallel AB$, cutting the circle at D and E. Then, by use of the diagonal scale, measure CD and CE. The numbers of units of length of CD and CE are the roots required. Prove it.



9. Solve the following quadratics by the method of Problem 8:

$$x^2 - 10x + 9 = 0; \quad x^2 - 15x + 25 = 0; \quad x^2 - 12x + 16 = 0;$$
$$x^2 - 18x + 49 = 0; \quad x^2 - 4x + 1 = 0.$$

10. Find the segments of a sect 20 in. long divided into extreme and mean ratio.

11. Given the longer part of a sect divided into extreme and mean ratio, construct the sect.

12. Experience has shown that a book, photograph, or other rectangular object is most pleasing to the eye when its length and width are obtained by dividing the half perimeter into extreme and mean ratio. Find to the nearest integer the width of such a book whose length is 8 in.

13. From a point P exterior to a given circle draw a secant cutting the circle at A and B, so that AB is a mean proportional between PA and PB.

14. Find a point P in the arc AB so that $\dfrac{PA}{PB} = \dfrac{3}{2}$.

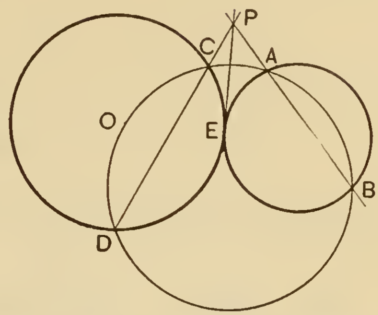
15. Through a given point P in the arc AB draw a chord that shall be bisected by chord AB.

16. Through one of the intersections of two given circles draw a line such that the two chords intercepted on it by the circle shall be in the ratio of 3 to 4.

17. Construct on one side of a triangle a point whose distances to the other two sides shall be proportional to two given sects.

18. Draw a circle to pass through two given points and be tangent to a given line. (See § 240.)

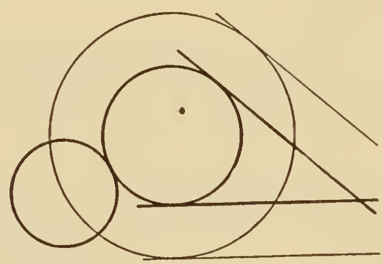
19. Draw a circle passing through two given points and tangent to a given circle.



Suggestion. — Let A and B be the given points and O the center of the given circle. Draw any circle through A and B, cutting given circle at C and D. Draw AB and CD, and let them meet at P. Draw PE tangent to the given circle.

20. Draw a circle through a given point and tangent to two given lines.

Suggestion. — Draw line through given point perpendicular to bisector of angle between given lines. On this line locate a second point at the same distance from the bisector that the given point is.



21. Draw a circle tangent to a given circle and tangent to two given lines.

Suggestion. — Make use of Ex. 20.

22. Construct two sects, having given their sum and their ratio.

CHAPTER X

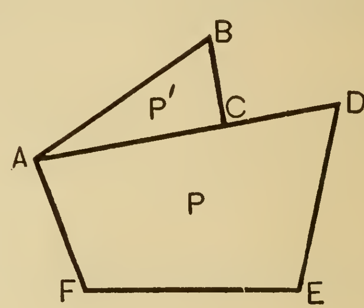
MENSURATION OF POLYGONS. COMPARISON OF AREAS

253. Areas. — The **area** of a polygon is the *numerical measure* of the surface inclosed by its sides, the *unit of measure* used being the surface of a square whose side is some unit of length.

Thus, if a rectangle can be divided into exactly 15 equal squares, and one of these squares is taken as the unit of measure, the *area* of the rectangle is 15.

Two polygons are said to be **equal** when they have equal areas. Hence, it is evident that *congruent polygons are necessarily equal, but equal polygons are not necessarily congruent.*

If two polygons are so arranged that they have a part of their perimeters in common, they are said to be **adjacent**. Thus, P and P', in the figure, are adjacent, AC being a part of each perimeter.



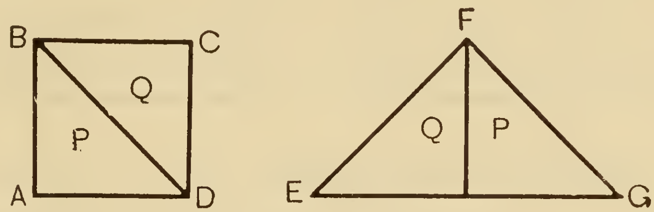
The adjacent polygons P and P' form another polygon $ABCDEF$, which is called the **sum** of P and P'. Likewise, P' is the **difference** between $ABCDEF$ and P.

It is evident that the area of the sum of two polygons is the sum of their areas, and that the area of the difference of two polygons is the difference of their areas.

It is clear, therefore, that *in the comparison of areas* in the following theorems, we may divide a polygon into any num-

ber of parts and rearrange these parts into any convenient figure, and the new polygon will be equal to the old one.

Thus, if the square *ABCD* is divided by *BD* into triangles *P* and *Q*, and *P* and *Q* are so placed as to form the triangle *EFG*, then the area of *EFG* equals the area of *ABCD*.

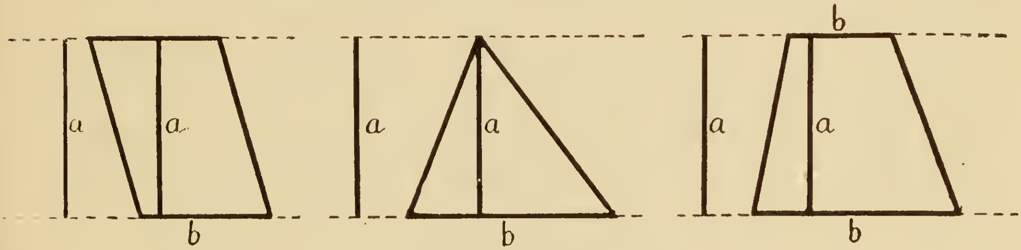


254. Definitions. — One side of a *parallelogram* is called the **base**, and the perpendicular distance between it and the opposite side is called the **altitude**.

In a *rectangle*, two adjacent sides may be taken as the **base** and **altitude**, since they are perpendicular.

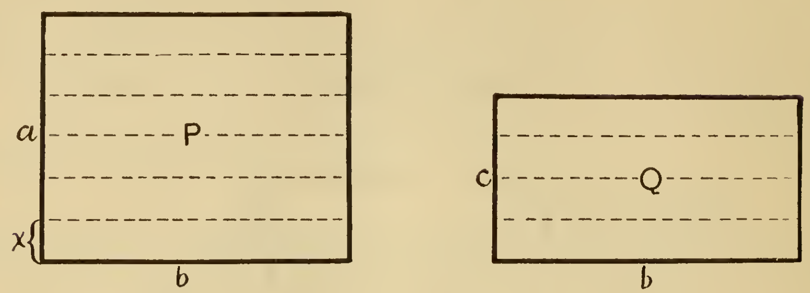
In a *triangle*, either side may be taken as the **base**, and the perpendicular distance between it and a line parallel to it through the opposite vertex is the **altitude**.

In a *trapezoid*, the two parallel sides are called **bases**, and the perpendicular distance between them the **altitude**.



Note. — In the work that follows, by "rectangle," "parallelogram," etc., is meant the "area of the rectangle," "area of the parallelogram," etc. Also, by "the product of two sects" is meant the product of their numerical measures.

255. Theorem. — *Two rectangles having equal bases are to each other as their altitudes.*



Case I. *Altitudes commensurable.*

Hypothesis. Rectangles P and Q have equal bases b, and commensurable altitudes a and c, respectively.

Conclusion. $\dfrac{P}{Q}=\dfrac{a}{c}.$

Proof. 1. Let x be a common measure of a and c.

2. Apply x to a and to c, and let it be contained m times in a and n times in c.

3. Through the points of division draw straight lines parallel to the bases.

4. Then P is divided into m congruent rectangles, and Q into n which are also congruent to those of P. Why?

5. $$\therefore \frac{P}{Q}=\frac{m}{n}.$$

6. But $$\frac{a}{c}=\frac{m}{n}.$$

7. $$\therefore \frac{P}{Q}=\frac{a}{c}.$$

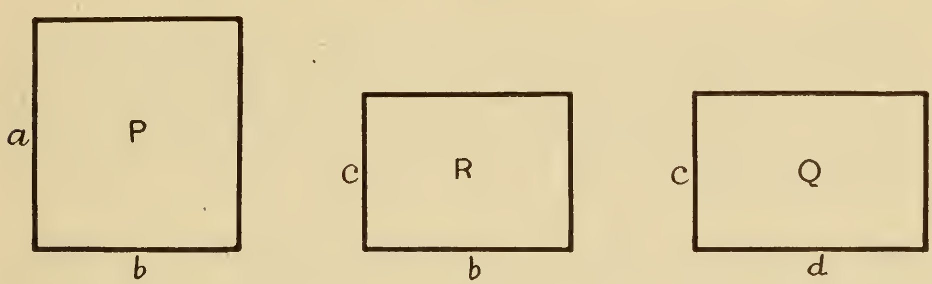
Case II, in which the altitudes are *incommensurable*, is not treated in this book, but the theorem is equally true in this case.

256. Corollary. — *Rectangles having equal altitudes are to each other as their bases.*

For any side may be taken as the base and the adjacent side as the altitude.

257. Remark. — The theorem in § 255 might have been stated: *Two rectangles having one dimension in one equal to one dimension of the other are proportional to the other dimensions.* This includes both the theorem and the corollary.

258. Theorem. — *Any two rectangles are to each other as the products of their bases and their altitudes.*



Hypothesis. Rectangles P and Q have altitudes and bases respectively, a and b, and c and d.

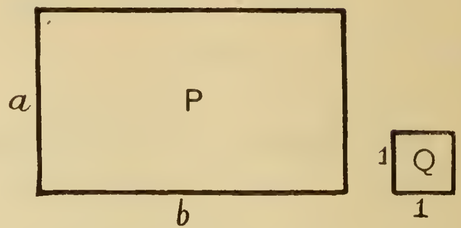
Conclusion. $\frac{P}{Q}=\frac{ab}{cd}.$

Proof. 1. Suppose a third rectangle R drawn with base equal to b and altitude equal to c.

2. Then $$\frac{P}{R}=\frac{a}{c}, \text{ and } \frac{R}{Q}=\frac{b}{d}.$$

3. $$\therefore \frac{P}{Q}=\frac{ab}{cd}.$$

259. Theorem. — *The area of a rectangle is equal to the product of its base and altitude.*



Hypothesis. P is a rectangle whose base is b and altitude a.

Conclusion. $P = ab$.

Proof. 1. Let Q be the unit area, *i.e.* a square whose side is 1.

2. $\frac{P}{Q} = \frac{a \cdot b}{1 \cdot 1} = ab.$

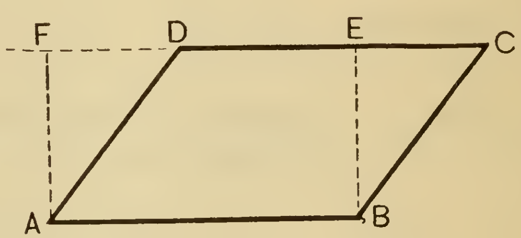
3. $\therefore P = ab \cdot Q.$

4. And since Q is the unit area, *i.e.* $Q = 1$, this becomes $P = ab$.

Remark. Since a rectangle is measured by the product of the numerical measures of its sides, some writers use the expression "the rectangle of two sects" instead of "the product of two sects."

260. Corollary. — *The area of a square is equal to the square of one of its sides.*

261. Theorem. — *Any parallelogram is equal to a rectangle having an equal base and an equal altitude.*



Hypothesis. $ABCD$ is a parallelogram with base AB and altitude EB.

Conclusion. $\square ABCD =$ a rectangle with base equal to AB and altitude equal to EB.

Proof. 1. Complete the rectangle $ABEF$ having this base and altitude.

2. Then $\triangle AFD \cong \triangle BEC$.

3. $\therefore$ quadrilateral $ABCF - \triangle AFD =$ quadrilateral $ABCF - \triangle BEC$.

4. That is, $\square ABCD =$ rectangle $ABEF$.

262. Theorem. — *The area of a parallelogram is equal to the product of its base and altitude.*

Suggestion. This follows from § 259 and § 261. Give complete proof.

263. Corollary 1. — *Parallelograms having equal bases and equal altitudes are equal.*

264. Corollary 2. — *Parallelograms are to each other as the products of their bases and altitudes.*

265. Corollary 3. — *Parallelograms having equal altitudes are to each other as their bases.*

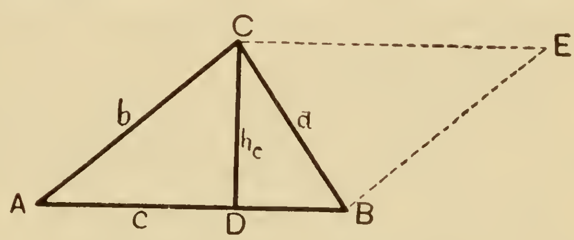
266. Corollary 4. — *Parallelograms having equal bases are to each other as their altitudes.*

267. Theorem. — *The area of any triangle is equal to one half the product of its base and altitude.*

Hypothesis. AB is the base and CD the altitude of $\triangle ABC$.

Conclusion. $\triangle ABC = \frac{1}{2} AB \cdot CD$.

Suggestion. Prove $\triangle ABC = \frac{1}{2} \square ABEC$.



268. Corollary 1. — *Triangles having equal bases and equal altitudes are equal.*

269. Corollary 2. — *Triangles are to each other as the products of their bases and altitudes.*

270. Corollary 3. — *Triangles having equal altitudes are to each other as their bases.*

271. Corollary 4. — *Triangles having equal bases are to each other as their altitudes.*

272. Computation. — *Compute the area of a triangle in terms of its sides.*

1. $\triangle ABC = \frac{1}{2} \cdot AB \cdot CD = \frac{1}{2} c \cdot h_c.$ (See § 267.)

2. But $h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$

3. $\therefore \triangle ABC = \frac{1}{2}c \cdot \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

273. Corollary. — $r = \dfrac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}},$

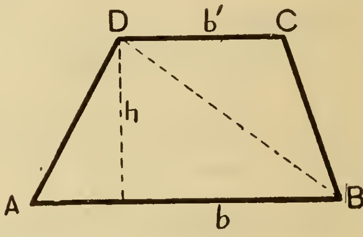
and $$\triangle ABC = \sqrt{s(s-a)(s-b)(s-c)}.$$

$$\therefore \triangle ABC = \frac{abc}{4r},$$

when $r =$ radius of the circumscribed circle.

274. Remark. — The *area of any polygon* may be found by decomposing it into triangles, if the areas of the triangles may be computed.

275. Theorem. — *The area of any trapezoid is equal to one half the product of its altitude and the sum of its bases.*



Hypothesis. $ABCD$ is a trapezoid with bases b and b' and altitude h.

Conclusion. $ABCD = \frac{1}{2} h(b + b').$

Suggestions. 1. Decompose $ABCD$ into triangles by drawing DB.

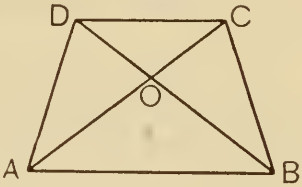
2. What is the altitude of $\triangle ABD$? of $\triangle DCB$?

3. What is the area of $\triangle ABD$? of $\triangle DCB$?

276. Corollary. — *The area of a trapezoid is equal to the product of its altitude and the sect joining the middle points of its non-parallel sides.*

EXERCISES

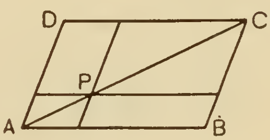
1. Find the area of an equilateral triangle in terms of one of its sides. See § 272.



2. Also find the area of an equilateral triangle in terms of its altitude.

3. The non-parallel sides of any trapezoid form with the diagonals two equal triangles.

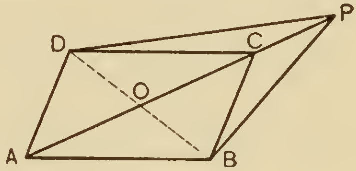
(To prove that $\triangle AOD = \triangle BOC$.)



4. Through any point P on the diagonal AC of the parallelogram $ABCD$, parallels are drawn to the four sides, forming parallelograms PD and PB. Prove parallelograms PD and PB equal.

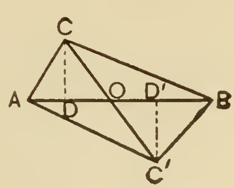
5. Any parallelogram is bisected by any straight line through the intersection of its diagonals.

6. Any trapezoid is bisected by the straight line joining the middle points of its parallel sides.



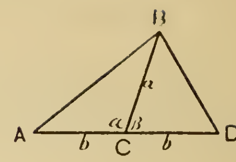
7. If P is any point on the diagonal AC produced, of parallelogram $ABCD$, then the triangles PCD and PCB are equal. (Compare $\triangle$ POD and POB, also $\triangle$ CDO and CBO.)

8. Is Exercise 7 true if P falls anywhere on AC between A and C? Prove the answer.



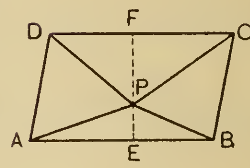
9. If two triangles are equal and have the same base but stand on opposite sides of it, prove that the sect joining their vertices is bisected by the base or the base produced.

Suggestion. — $\triangle CDO \cong \triangle C'D'O$.



10. Two triangles are equal if two sides of one are equal to two sides of the other, and the included angles supplementary.

(Place them as in the figure.)



11. If a point within a parallelogram is joined to the four vertices, the sum of the two triangles thus formed having for bases two parallel sides is equal to one half the parallelogram.

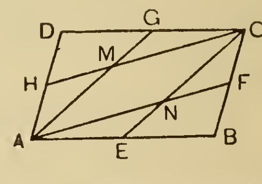
(To prove that $\triangle PAB + \triangle PCD = \frac{1}{2} \square ABCD$.)

12. Discuss Exercise 11 when P is in one of the sides.

13. Is the theorem, Ex. 11, true when P is without the parallelogram? If not, what relation does exist between the triangles thus formed and the parallelogram? Prove your answer.

14. The area of a rhombus is equal to half the product of its diagonals.

15. If from two opposite vertices of any parallelogram sects are drawn to the middle points of the sides, the parallelogram thus formed is equal to one third of the given parallelogram.



(To prove $\square ANCM = \frac{1}{3} \square ABCD$.)

Suggestions.—1. Prove $\square AECG = \frac{1}{2} ABCD$.

2. Then prove $\square ANCM = \frac{2}{3} \square AECG$.

16. The sects joining the middle points of the sides of any triangle divide the triangle into four equal triangles.

17. The triangle having one of the non-parallel sides of a trapezoid as base and the middle point of the opposite side as vertex is equal to one half of the trapezoid.

18. If from the middle point of either diagonal of any quadrilateral sects be drawn to the two opposite vertices, they divide the quadrilateral into two equal quadrilaterals.

19. If from the point of intersection of the medians of any triangle sects are drawn to the three vertices of the triangle, they form with the sides three equal triangles.

20. If from the middle point of any side of a triangle straight lines are drawn parallel to the other two sides, the parallelogram thus formed is equal to one half of the triangle.

21. Two isosceles triangles having the altitude of one equal to one half the base of the other, and their sides equal each to each, are equal.

22. If the middle points of the adjacent sides of any parallelogram be joined, the four triangles thus formed are equal, and together they are equal to the parallelogram thus formed by the lines joining the middle points.

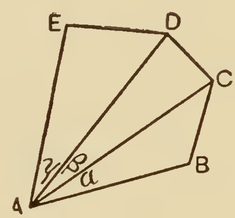
23. The area of a trapezoid is equal to the product of one of its nonparallel sides and the distance to it from the middle point of the opposite side. (See Exercise 17.)

24.* Prove that in a triangle of which the sides a and b include C, the area equals $\frac{1}{2}ab \sin C$.

Suggestion. — Draw a perpendicular to b from the extremity of a.

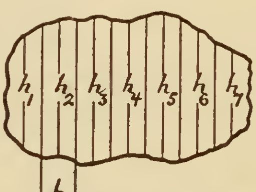
25.* Find the area of a triangle with two sides 40 ft. and 60 ft., respectively, and the included angle 54°.

26.* To find the area of a field $ABCDE$, the surveyor measured with the chain the lengths of the sides AB and AE and of the diagonals AC and AD, and measured the angles α, β, and γ. He found that $AB = 300$ yd., $AC = 520$ yd., $AD = 500$ yd., $AE = 400$ yd., $\alpha = 20°$, $\beta = 30°$, and $\gamma = 35°$. Compute the area of the field.



27.* Measure some field in the neighborhood and compute its areas as in Problem 26.

28. A fairly accurate method used for computing the area bounded by an irregular curve, known as the "mean ordinate" method, is as follows: At equal distances b measure the widths h_1, h_2, h_3, etc., of the area inclosed. Show that, if a large number of widths are measured, a close approximation to the area is given by



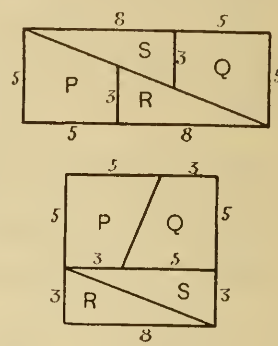
$$\text{area of figure} = (h_1 + h_2 + h_3 + \cdots)b.$$

29. The area of a triangle equals one half the product of its perimeter and the radius of the inscribed circle.

Suggestion. — Join the center to each of the vertices.

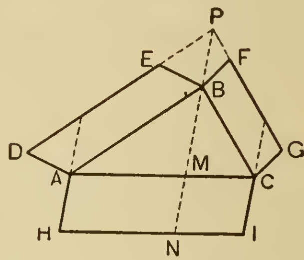
30. The papyrus of Ahmes, the oldest mathematical book extant, written by the Egyptian about 1700 B.C., gave for an isosceles triangle whose sides are 10 *ruths* and base 4 *ruths*, the area of 20 sq. ruths. By what rule must this area have been computed? What is the true area?

31. One base of a trapezoid is 10 ft., the altitude 4 ft., and the area 32 sq. ft. Find the length of a sect drawn between the non-parallel sides, parallel to the base, and 1 ft. from it.



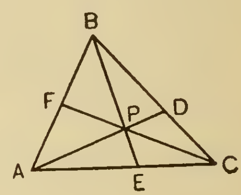
32. Explain the fallacy in the following popular construction: A piece of paper 8 in. square is cut into four pieces, P, Q, R, S, as shown in the figure. These pieces are then placed in new positions so as to form, apparently, a rectangle whose area is 65 sq. in.

Suggestion. — Use similar triangles.



33. Upon two sides of any triangle ABC the parallelograms $ABED$ and $BCGF$ are drawn. DE and GF are produced to meet at P. PB is drawn. Parallelogram $AHIC$ is drawn with side AH equal and parallel to PB. Prove that $\square\, AHIC = \square\, ABED + \square\, BCGF$.

Suggestion. — Show that $\square\, ABED = \square\, AHNM$, and $\square\, BCGF = \square\, MNIC$.



34. The lines AD, BE, and CF are drawn from the vertices of $\triangle ABC$ so that they meet at a point P. Show that

$$\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = 1.$$

Suggestion. —

$$\frac{BD}{DC} = \frac{\triangle APB}{\triangle APC};\ \frac{CE}{EA} = \frac{\triangle BPC}{\triangle APB};\ \frac{AF}{FB} = \frac{\triangle APC}{\triangle BPC}. \qquad \text{Why?}$$

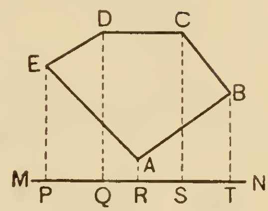
35. Find the area of a triangle whose sides are 6, 8, and 12; of one whose sides are 15, 16, and 17; of one each of whose sides is 10.

36. How many pieces of sod will it take to sod a lawn 42 ft. wide and 56 ft. long, if the pieces are 12 in. by 14 in.?

37. On a map drawn to scale of 60 mi. to the inch, what area is inclosed in a strip 3 in. wide and 5 in. long?

38. A canal is being excavated that is 28 ft. deep, 120 ft. wide at the top, and 90 ft. wide at the bottom. What is the area of a cross section of it?

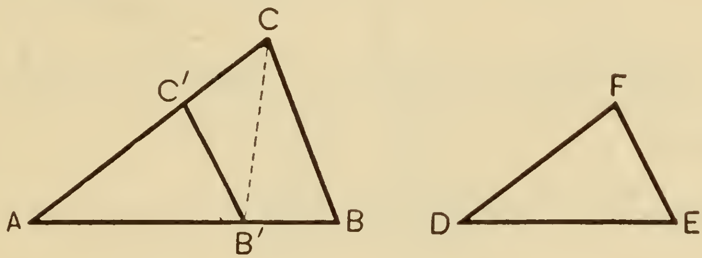
39. Surveyors sometimes find the area of a tract of land as follows: A base line MN is staked off east and west, say. From the various points in the boundary, the distances to this line are then measured, as EP, DQ, etc., and the distances PQ, QS, etc., are measured. Show how to compute the area of $ABCDE$.



If $EP = 2000$ ft., $DQ = 3000$ ft., $CS = 3000$ ft., $BT = 1500$ ft., $AR = 400$ ft., $PQ = 800$ ft., $QS = 1200$ ft., $ST = 1500$ ft., $TR = 2000$ ft., compute the area.

COMPARISON OF AREAS

277. Theorem. — *Two triangles having an angle of one equal to an angle of the other are to each other as the products of the sides including these angles.*



Hypothesis. In $\triangle ABC$ and DEF, $\angle A = \angle D$.

Conclusion. $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AB \cdot AC}{DE \cdot DF}$.

Proof. 1. Place $\triangle DEF$ upon $\triangle ABC$ so that $\angle D$ coincides with $\angle A$, and so that its position will be $AB'C'$, as in the figure.

2. Draw CB'.

3. Then $\dfrac{\triangle ABC}{\triangle AB'C} = \dfrac{AB}{AB'}$ and $\dfrac{\triangle AB'C}{\triangle AB'C'} = \dfrac{AC}{AC'}$. (§ 270)

4. $\therefore \dfrac{\triangle ABC}{\triangle AB'C'} = \dfrac{AB \cdot AC}{AB' \cdot AC'}$. (By multiplication)

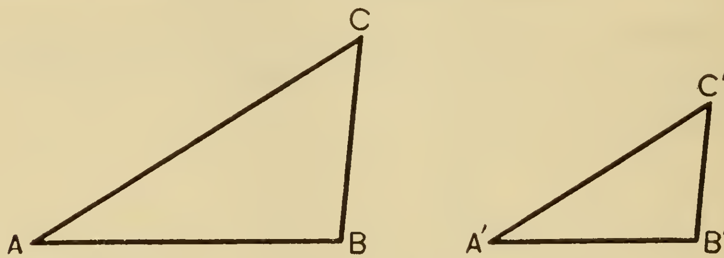
5. $\therefore \dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AB \cdot AC}{DE \cdot DF}$. ($\because \triangle AB'C' \cong \triangle DEF$)

EXERCISES

1. Two triangles having an angle of one supplementary to an angle of the other are to each other as the products of the sides including these angles.

2. Two parallelograms with equal angles are to each other as the products of the sides including a pair of equal angles.

278. Theorem. — *Two similar triangles are to each other as the squares of any two homologous sides.*



Hypothesis. $\triangle ABC \sim \triangle A'B'C'$.

Conclusion. $$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

Proof. 1. $\angle A = \angle A'$.

2. $$\therefore \frac{\triangle ABC}{\triangle A'B'C'} = \frac{AB \cdot AC}{A'B' \cdot A'C'} = \frac{AB}{A'B'} \cdot \frac{AC}{A'C'}.$$

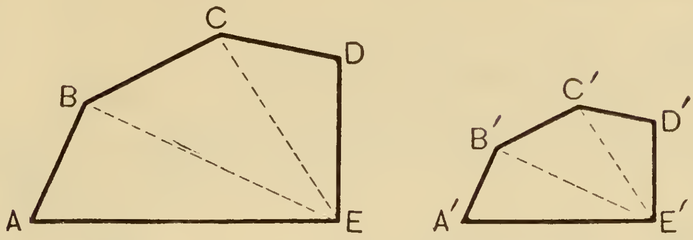
3. But $$\frac{AC}{A'C'} = \frac{AB}{A'B'}.$$

4. $$\therefore \frac{\triangle ABC}{\triangle A'B'C'} = \frac{AB}{A'B'} \cdot \frac{AB}{A'B'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

279. Theorem. — *Two similar polygons are to each other as the squares of any two homologous sides.*

Hypothesis. Polygons $ABCD \cdots$ and $A'B'C'D' \cdots$ are similar, with AB and $A'B'$ as homologous sides.

Conclusion. $\dfrac{ABCD\cdots}{A'B'C'D'\cdots}=\dfrac{\overline{AB}^2}{\overline{A'B'}^2}.$



Proof. 1. Draw diagonals of each, dividing the polygons into similar triangles.

2. $$\frac{\triangle ABE}{\triangle A'B'E'}=\frac{\overline{AB}^2}{\overline{A'B'}^2};$$

$$\frac{\triangle BCE}{\triangle B'C'E'}=\frac{\overline{BC}^2}{\overline{B'C'}^2}=\frac{\overline{AB}^2}{\overline{A'B'}^2};\ \text{etc.}$$

3. $$\therefore \triangle ABE=\frac{\overline{AB}^2}{\overline{A'B'}^2}\cdot\triangle A'B'E';$$

$$\triangle BCE=\frac{\overline{AB}^2}{\overline{A'B'}^2}\cdot\triangle B'C'E';\ \text{etc.}$$

4. $\therefore \triangle ABE+\triangle BCE+\cdots$

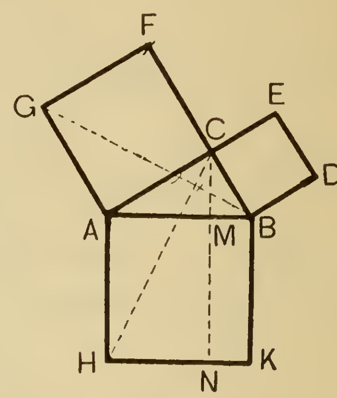
$$=\frac{\overline{AB}^2}{\overline{A'B'}^2}\left(\triangle A'B'E'+\triangle B'C'E'+\cdots\right).$$

5. $$\therefore \frac{\triangle ABE+\triangle BCE+\cdots}{\triangle A'B'E'+\triangle B'C'E'+\cdots}=\frac{\overline{AB}^2}{\overline{A'B'}^2};$$

or, $$\frac{ABCD\cdots}{A'B'C'D'\cdots}=\frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

Exercise. — The areas of two similar polygons are 324 sq. ft. and 576 sq. ft., respectively. Compare their homologous sides.

280. Theorem. — *The square described upon the hypotenuse of a right triangle is equal to the sum of the squares described upon the two legs.*



Hypothesis. In rt. $\triangle ABC$, AK is the square upon the hypotenuse, and BE and AF are the squares upon the legs.

Conclusion. $AK = BE + AF$.

Proof. 1. Draw GB and CH, and draw $CN \perp HK$.

2. $\angle GAB = \angle CAH$ (?), $GA = CA$, and $AB = AH$.

3. $\therefore \triangle GAB \cong \triangle CAH$.

4. But $\triangle GAB = \frac{1}{2}$ square AF.

5. And $\triangle CAH = \frac{1}{2}$ rectangle AN.

6. $\therefore$ square $AF =$ rectangle AN.

7. Likewise, square $BE =$ rectangle BN.

8. $\therefore$ adding in steps 6 and 7, $AK = AF + BE$.

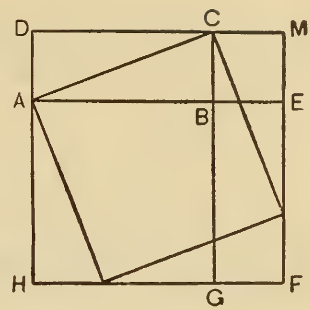
Note. — This theorem was proved in § 233, for the squares AK, AF, and BE have for their numerical measures $\overline{AB}^2$, $\overline{AC}^2$, and $\overline{CB}^2$, respectively (§ 260). The proof given here is practically the same as the one found in Euclid's "Elements," written about 250 or 300 B.C.

EXERCISES ON THE PYTHAGOREAN THEOREM

1. In the figure of § 280, prove $CH \perp GB$. (Let the intersection of CH and GB be O, and of CA and GB be L. Prove $\triangle GAL \sim \triangle COL$.)

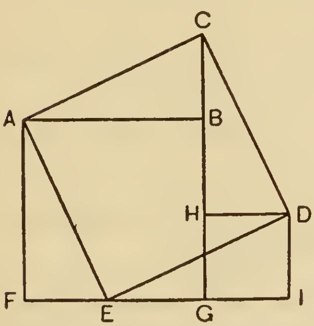
2. A large number of proofs have been worked out for the Pythagorean theorem. On the next page are given figures for some of the most interesting ones. Give the proofs suggested by these figures.

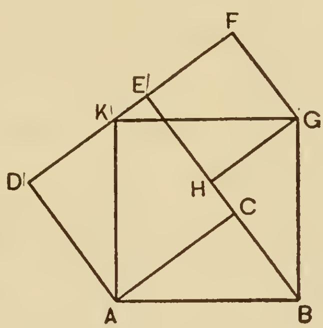
If the rectangles DB and BF be taken from the square HM, what remains? If the 4 congruent $\triangle$ at the corners are taken away, what remains? Then compare the $\triangle$ with the rectangles.

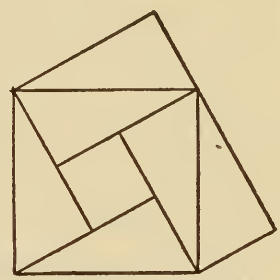


(This may be the proof given by Pythagoras.)

(Bhaskara was a native of India, about 1150 A.D., who wrote on mathematics.)







Bhaskara's Proof.

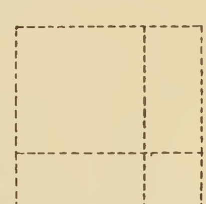
EXERCISES

1. The parallelogram formed by joining the middle points of the adjacent sides of any quadrilateral is equal to one half of the quadrilateral.

2. In any rectangle, the sum of the squares on the sects from any point within the rectangle to two opposite vertices is equal to the sum on the squares of the sects to the other vertices.

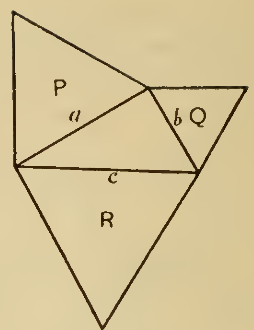
SUGGESTION. — Through the point draw lines parallel to the sides.

3. The square upon the *sum* of two sects is equal to the sum of the squares upon the sects, *plus* twice the rectangle formed by the sects as adjacent sides.



4. The square upon the *difference* of two sects is equal to the sum of the squares upon the sects, *less* twice the rectangle formed by the sects.

5. The rectangle formed by the sum and the difference of two sects is equal to the difference of the squares upon the two sects.



6. The equilateral triangle described upon the hypotenuse of a right triangle is equal to the sum of the equilateral triangles described upon the two legs.

SUGGESTION. — $\frac{P}{R} = \frac{a^2}{c^2}$; $\frac{Q}{R} = \frac{b^2}{c^2}$; $a^2 + b^2 = c^2$.

7. If similar polygons are constructed upon the sides of a right triangle as homologous sides, the polygon upon the hypotenuse is equal to the sum of the polygons upon the other two sides.

8. The areas of two similar triangles are to each other as the squares of the corresponding altitudes.

9. If one of two similar polygons is double the other, what is the ratio of their homologous sides?

10. A line is drawn parallel to the base of a triangle and dividing it into two equal parts. It cuts off what part of the altitude?

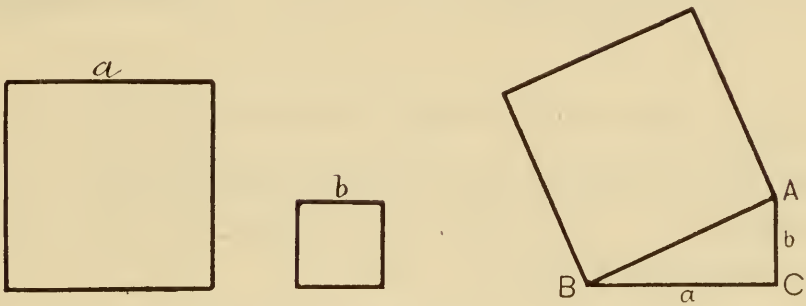
11. The altitude of a triangle is 20 in. A line parallel to the base and 12 in. from the base cuts off a triangle that is what part of the given triangle?

12. The homologous sides of two similar polygons are 6 in. and 12 in., respectively. Find the homologous side of a similar polygon equal to their sum.

CHAPTER XI

CONSTRUCTIONS

281. Construction. — *Construct a square equal to the sum of two given squares.*

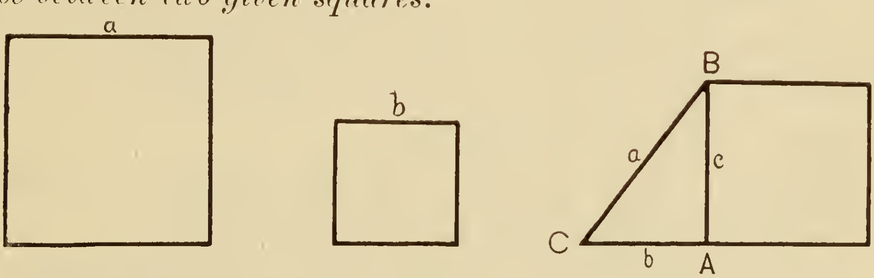


Given two squares whose sides are a and b, respectively.

Required to construct a square equal to the sum of the squares on a and b.

(Theorem § 280 suggests the construction.)

282. Construction. — *Construct a square equal to the difference between two given squares.*

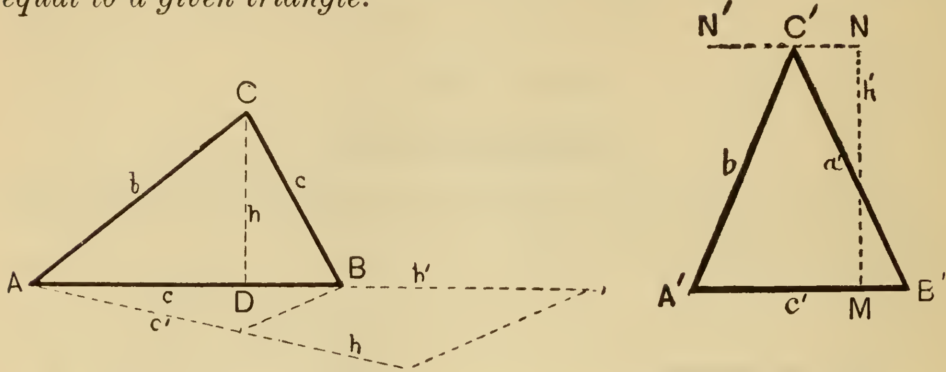


Given two squares whose sides are a and b, respectively.

Required to construct a square equal to the difference between the squares on a and b.

(Construction and proof left to the student.)

283. Construction. — *Construct, on a given base, a triangle equal to a given triangle.*



Given $\triangle ABC$ with the base c and altitude h, also $A'B' = c'$.

Required to construct on c' as base a triangle equal to ABC.

Analysis. 1. If we let h' = the altitude of the required triangle, then $c'h' = ch$.

2. $\therefore \dfrac{c'}{c} = \dfrac{h}{h'}$, or h' is the fourth proportional to c', c, and h.

Construction. 1. Construct the fourth proportional to c', c, and h. Call it h'.

2. Erect $\perp$ to $A'B'$ from any point M, and lay off $MN = h'$.

3. Through N draw $NN' \parallel A'B'$.

4. From any point C' in NN' draw sects to A' and B', and the triangle $A'B'C'$ will be the one required.

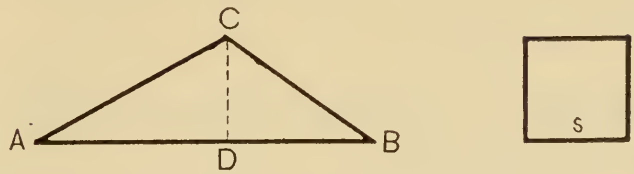
Proof. (Left to the student.)

Discussion. Show that an infinite number of triangles can be drawn fulfilling the required conditions.

Name some additional condition that, if imposed upon the required triangle, would give a single solution.

284. Construction. — *Construct a triangle on a given base and equal to a given parallelogram.*

285. Construction. — *Construct a square equal to a given triangle.*



Given $\triangle ABC$ with base AB and altitude CD.

Required to construct a square equal to $\triangle ABC$.

Analysis. 1. Let s be the side of the required square.

2. Then $s^2 = \frac{1}{2} AB \cdot CD$.

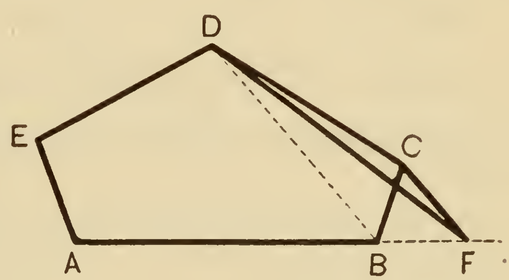
3. $\therefore s$ is the mean proportional between AB and $\frac{1}{2} CD$ (or $\frac{1}{2} AB$ and CD).

(Let the student complete the construction and proof.)

286. Construction. — *Construct a triangle equal to a given polygon.*

Given polygon $ABCDE$.

Required to construct a triangle equal to $ABCDE$.



Analysis. 1. $ABCDE$ may be transformed into an equal triangle by successive steps in which the given polygon is replaced each time by a polygon having the number of sides less by one.

2. If AB is produced to F, and DF drawn, then polygon $AFDE$ will equal $ABCDE$ if $CF \parallel DB$, because $\triangle BFD = \triangle BCD$, and hence $ABDE + \triangle BFD = ABDE + \triangle BCD$.

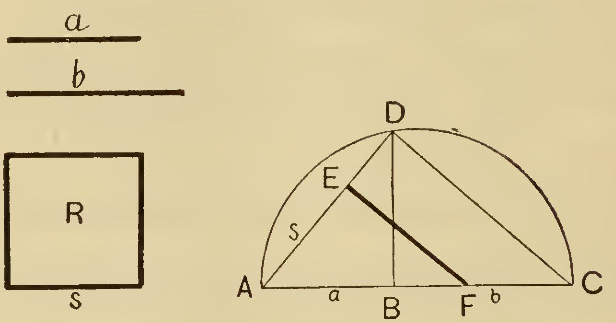
3. This gives a polygon equal to $ABCDE$ and with the number of sides one less, and a repetition of the process will finally lead to a triangle.

Construction. (Left to the student.)

287. Construction. — *Construct a square equal to any given polygon.*

(Use § 285 and § 286.)

288. Construction. — *Construct a square that shall be to a given square in a given ratio.*



Given square R with side s, and sects a and b.

Required to construct a square X such that $\frac{R}{X} = \frac{a}{b}$.

Analysis. Theorems §§ 232 and 260 suggest a construction.

Construction. 1. On a straight line lay off $AB = a$ and $BC = b$.

2. Describe ⊙ on AC as diameter, and from B erect a ⊥ cutting ⊙ at D.

3. Draw AD and CD.

4. Now $\frac{\overline{AD}^2}{\overline{DC}^2} = \frac{a}{b}$.

5. ∴ on AD lay off $AE = s$, and through E draw $EF \parallel DC$ and cutting AC in F.

6. Now $\frac{s}{EF} = \frac{AD}{DC}$.

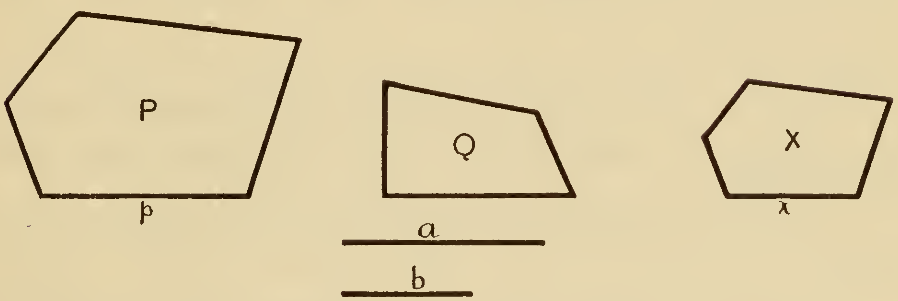
7. ∴ EF is the side of the required square.

Proof. 1. $\frac{s^2}{\overline{EF}^2} = \frac{\overline{AD}^2}{\overline{DC}^2}$.

2. But $\frac{\overline{AD}^2}{\overline{DC}^2} = \frac{a}{b}$.

3. $\therefore \frac{s^2}{\overline{EF}^2} = \frac{a}{b}$.

289. Construction.—*Construct a polygon similar to one of two given polygons and equal to the other.*



Given polygons P and Q.

Required to construct a polygon X similar to P and equal to Q.

Analysis. 1. Suppose x to be the side of the required polygon homologous to p of P.

2. Then $\frac{P}{X} = \frac{p^2}{x^2}$.

3. Let a be the side of a square equal to P and b be the side of a square equal to Q.

4. Then $\frac{P}{Q} = \frac{a^2}{b^2}$. And $\because X = Q$, $\therefore \frac{P}{X} = \frac{a^2}{b^2}$.

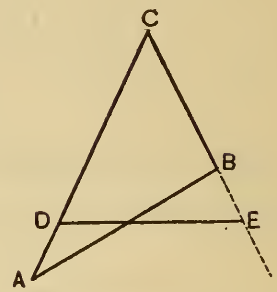
5. $\therefore \frac{a}{b} = \frac{p}{x}$, *i.e.* x is the fourth proportional to a, b, and p.

(Let the student give the construction and proof.)

EXERCISES

1. Construct an isosceles triangle equal to a given triangle and on the same base.

2. Construct a square equal to a given parallelogram.



3. Construct an isosceles triangle whose vertical angle is one of the angles of a given triangle, and whose area is equal to that of the given triangle.

SUGGESTION. — $\overline{CD}^2 = CA \cdot CB$.

4. Construct a square equal to the sum of two given polygons.

SUGGESTION. — First construct squares equal to the given polygons.

5. Construct a square equal to the sum of a given square and a given triangle.

6. Construct a square equal to the sum of any number of given squares.

7. Construct a polygon similar to two given similar polygons and equal to their sum. To their difference.

8. Construct a right triangle having a given altitude and equal to a given parallelogram.

9. Construct an equilateral triangle equal to a given triangle.

10. Construct an equilateral triangle equal to a given polygon.

11. Construct a triangle having a given angle and equal to a given parallelogram.

12. Construct a parallelogram equal to a given triangle and having one of its angles equal to a given angle.

13. Construct a rectangle having a given side and equal to a given square; equal to a given rectangle; equal to a given parallelogram.

14. Construct a polygon similar to a given polygon and having a given ratio to it.

15. Construct a triangle equal to a given trapezoid and having for its base the longer base of the trapezoid.

16. Construct a rhombus having a given diagonal and equal to a given parallelogram.

Suggestion. — The diagonals of a rhombus are perpendicular.

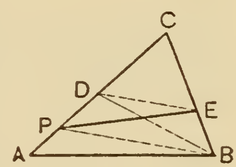
17. Through a given point draw a straight line bisecting a given parallelogram.

Suggestion. — Where must the line cut a diagonal?

18. Draw a straight line parallel to a given straight line and bisecting a given parallelogram.

19. Bisect a triangle by a straight line drawn through any vertex.

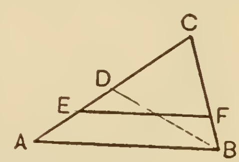
20. Divide a triangle into any given ratio by a straight line drawn through any vertex.



21. Bisect a triangle by drawing a straight line through any given point on a side of the triangle. Such a problem is sometimes met in land surveying.

Analysis. — Let P be the given point. $\because$ a sect from B to D, the middle point of AC, bisects the triangle, $\therefore$ it follows that a point E must be found on BC so that $\triangle PEB = \triangle PDB$.

22. Trisect a triangle by drawing straight lines through any given point on one of the sides.



23. Bisect a triangle by a straight line parallel to one of the sides.

Analysis. — 1. Suppose the problem solved and that EF is the line.

2. The median BD bisects the triangle.

3. $\therefore \triangle DCB = \triangle ECF.$

4. $\therefore CE \cdot CF = CD \cdot CB.$

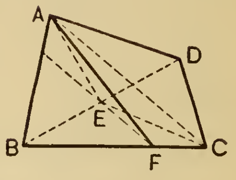
5. Also $\dfrac{CE}{CA} = \dfrac{CF}{CB}.$

6. And from 4, $\dfrac{CD}{CE} = \dfrac{CF}{CB}.$

7. $\therefore \dfrac{CD}{CE} = \dfrac{CE}{CA}$, *i.e.* CE is the mean proportional between CD and CA.

Give the construction and proof.

24. Bisect any quadrilateral by a straight line drawn through one of the vertices.

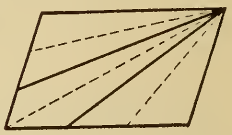


Analysis. — 1. Let A be the vertex.

2. If E is the middle point of BD, then AE and CE divide the quadrilateral into equal parts.

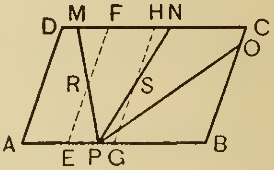
3. $\therefore$ It remains to construct a triangle equal to $\triangle AEC$, having for base AC and a vertex in one of the sides of the quadrilateral.

25. Trisect any quadrilateral by straight lines drawn through any vertex.



26. Divide a parallelogram into any number of equal parts by lines drawn from one of its vertices.

27. Divide a parallelogram into four equal parts by drawing lines through any point in one of the sides.



Suggestion. — Let P be the given point. Divide AB into four equal parts. Draw EF and GH parallel to AD. Bisect EF at R and GH at S. Having drawn PM and PN, construct PO by Ex. 24.

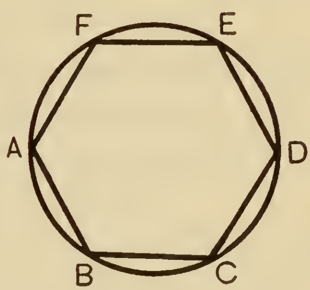
CHAPTER XII

REGULAR POLYGONS; CIRCLES

290. Definitions. — A **regular polygon** is one that is both equilateral and equiangular.

A polygon of five sides is a **pentagon**; one of six sides, a **hexagon**; one of eight sides, an **octagon**; one of ten sides, a **decagon**; one of fifteen sides, a **pentedecagon**.

291. Theorem. — *An equilateral polygon inscribed in a circle is regular.*



Hypothesis. In the inscribed polygon $ABC \cdots$, $AB = BC = CD =$ etc.

Conclusion. The polygon $ABC \cdots$ is regular, *i.e.* angles A, B, C, etc., are equal.

Suggestion. Show that these angles are measured by equal arcs.

292. Theorem. — *A circle may be circumscribed about, or inscribed in, any regular polygon.*

Hypothesis. The polygon $ABC \cdots$ is regular.

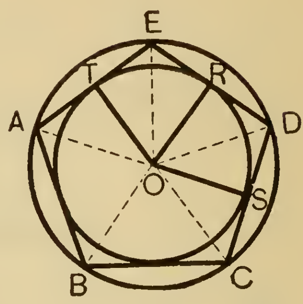
Conclusion. We are to prove two theorems, viz:

(I) A circle may be circumscribed about $ABC \cdots$, or

(II) A circle may be inscribed in $ABC \cdots$.

Proof. (I) 1. Draw a circle through three vertices, A, B, C. Let its center be O.

2. Draw radii OA, OB, and OC. Also draw sect OD.



3. Now $\angle CBA = \angle DCB$ (Why?).

4. Also $\angle CBO = \angle OCB$, for $\triangle OBC$ is isosceles.

5. Hence, $\angle OBA = \angle DCO$, by subtraction.

6. $\therefore \triangle OAB \cong \triangle OCD$ (Why?).

7. $\therefore OD = OA$, *i.e.* OD is a radius.

8. Hence the circle passes through D.

In the same way the circle may be shown to pass through all the vertices.

Proof. (II) 1. AB, BC, etc., are equal chords of the circumscribed circle.

2. Hence, AB, BC, etc., are equally distant from O, *i.e.* the perpendiculars OR, OS, etc., to the chords are equal.

3. $\therefore$ a circle with center O and radius equal to OR will pass through the points R, S, T, etc.

4. $\therefore$ the sides of the polygon are tangents to this circle (§ 173), and hence the circle is inscribed in the polygon.

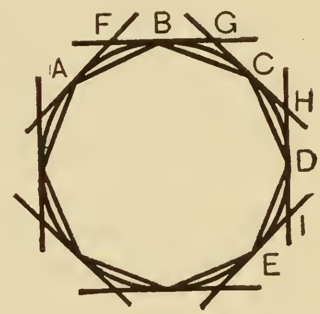
293. Corollary 1. — *The inscribed and circumscribed circles of a regular polygon have the same center.*

294. The parts of a regular polygon. — The **center** of a regular polygon is the common center of the inscribed and circumscribed circles. The **radius** of a regular polygon is the radius of the circumscribed circle. The **apothem** of the polygon is the radius of the inscribed circle. The **angle at the center** is the angle between the radii drawn to the extremities of any side.

295. Corollary 2. — *The angle at the center of any regular polygon is equal to four right angles divided by the number of sides of the polygon.*

296. Corollary 3. — *The radius drawn to any vertex of a regular polygon bisects the angle at the vertex.*

297. Theorem. — *If any circle be divided into any number of equal arcs,* (1) *the chords of these arcs form a regular inscribed polygon; and* (2) *the tangents to the circle at the points of division form a regular circumscribed polygon.*



Suggestions. (1) Prove $AB = BC$ = etc.; and (2) 1. Prove AFB, BGC, etc., congruent isosceles triangles. 2. Then prove that $FGH \cdots$ is equilateral and equiangular.

298. Corollary 1. — *If the vertices of a regular inscribed polygon are joined to the middle points of the arcs subtended by the sides of the polygon, the joining sects will form another regular inscribed polygon of twice the number of sides.*

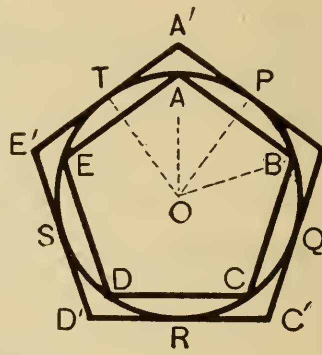
299. Corollary 2. — *The tangents to a circle at the middle points of the arcs subtended by the sides of a regular inscribed polygon form a circumscribed regular polygon of double the number of sides.*

300. Corollary 3. — *The perimeter of an inscribed regular polygon is less than that of an inscribed regular polygon of double the number of sides.*

Suggestion. The sum of two sides of a triangle is greater than the third side.

301. Corollary 4. — *The perimeter of a circumscribed regular polygon is greater than that of a circumscribed regular polygon of double the number of sides.*

302. Theorem. — *The tangents to a circle at the middle points of the arcs subtended by the sides of a regular inscribed polygon form a regular circumscribed polygon whose sides are parallel to the corresponding sides of the inscribed polygon.*

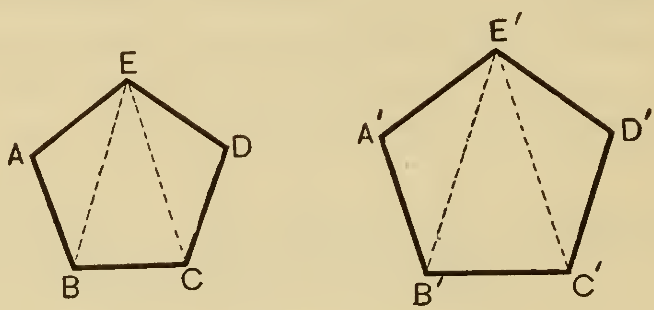


Suggestions. 1. Prove arcs PQ, QR, etc., equal, and use § 297.

2. Now prove AB and $A'B'$ both $\perp OP$, hence parallel.

303. Theorem. — *Two regular polygons of the same number of sides are similar.*

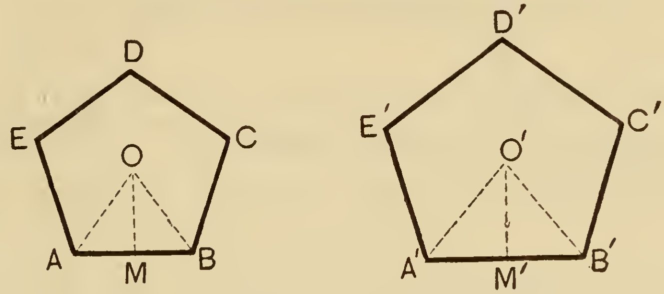
Suggestion. Prove that they may be divided into the same number of triangles similar each to each and similarly placed.



304. Corollary. — *The areas of two regular polygons of the same number of sides are to each other as the squares of any two homologous sides* (§ 279).

305. Theorem. — *The perimeters of two regular polygons of the same number of sides are to each other as their radii, and also as their apothems.*

Hypothesis. $ABC \cdots$ and $A'B'C' \cdots$ are regular polygons having the same number of sides. P and P', and O and O' are their perimeters and centers, respectively. Also R and R' are their radii, and r and r' their apothems, respectively.



Conclusion. $$\frac{P}{P'} = \frac{R}{R'} = \frac{r}{r'}.$$

Suggestions. 1. $$\frac{P}{P'} = \frac{AB}{A'B'}.$$

2. Show that $\triangle AOB \sim \triangle A'O'B'$, and $\triangle AOM \sim \triangle A'O'M'$.

3. Then show that $$\frac{AB}{A'B'} = \frac{OA}{O'A'} = \frac{OM}{O'M'}.$$

4. Hence, $$\frac{P}{P'} = \frac{OA}{O'A'} = \frac{OM}{O'M'};$$

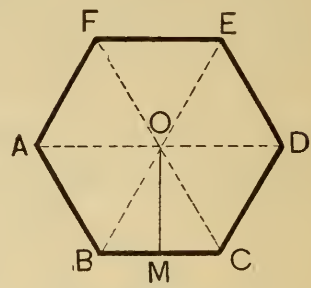
5. or, $$\frac{P}{P'} = \frac{R}{R'} = \frac{r}{r'}.$$

306. Corollary. — *The areas of two regular polygons of the same number of sides are to each other as the squares of their radii and of their apothems.*

Suggestion. Use § 304 and the fact from algebra that if

$$\frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a^2}{b^2} = \frac{c^2}{d^2}.$$

307. Theorem. — *The area of any regular polygon is equal to one half the product of its perimeter by its apothem.*



Hypothesis. $ABC \cdots$ is a regular polygon, with area S, perimeter P, and apothem r.

Conclusion. $S = \frac{1}{2} P r$.

Suggestions. 1. Draw the radii of the polygon, dividing it into as many triangles as it has sides.

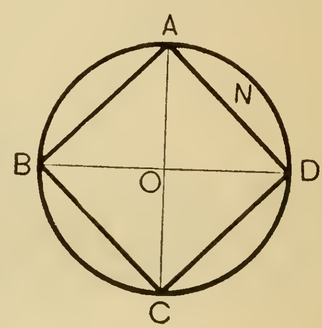
2. Now since the altitude of each triangle is r $(= OM)$, show that the sum of the areas of the triangles is

$$\tfrac{1}{2}(AB + BC + \cdots)r.$$

INSCRIPTION AND CIRCUMSCRIPTION OF REGULAR POLYGONS

308. Construction. — *Inscribe a square in a circle.*

Analysis. ∵ a square is a regular polygon of four sides, it is necessary to divide the circle into quadrants, which can be done by drawing two perpendicular diameters.

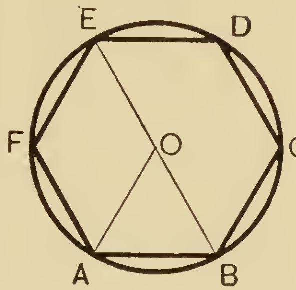


Construction. (Left to the student.)

309. Corollary 1. — *A regular polygon of 4, 8, 16, $\cdots 2^n$ sides may be inscribed in a circle.* How?

310. Corollary 2. — *By drawing tangents at the vertices of regular inscribed polyons of 4, 8, 16, $\cdots 2^n$ sides, regular circumscribed polygons may be constructed with the same numbers of sides.* (See § 298.)

311. Construction. — *Inscribe a regular hexagon in a circle.*



Analysis. 1. ⚠ formed by joining the center to the vertices of the polygon are isosceles.

2. The central angle of a regular hexagon is 60°.

3. ∴ the triangles are equilateral.

4. ∴ the sides of the hexagon must each equal a radius.

Construction. (Left to the student.)

312. Corollary 1. — *An equilateral triangle may be inscribed in a circle.* How? Inscribe one.

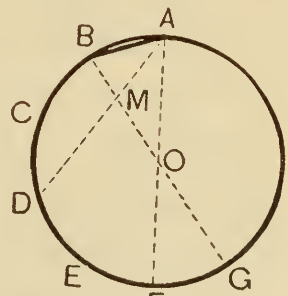
313. Corollary 2. — *Regular polygons of* 3, 6, 12, 24, $\cdots 3 \cdot 2^n$ *sides may be inscribed in a circle.*

How? Inscribe a regular polygon of 12 sides; one of 24.

314. Corollary 3. — *By drawing tangents at the vertices of regular inscribed polygons with* 3, 6, 12, $\cdots 3 \cdot 2^n$ *sides, regular circumscribed polygons may be constructed with the same numbers of sides.*

315. Construction. — *Inscribe a regular decagon in a circle.*

Analysis. 1. Suppose the problem solved and that A, B, C, D, $\cdots$ are the vertices.



2. Draw AF and BG. These lines will pass through O. (Why?)

3. Join A and D.

4. $\triangle OMA$ is isos. (Why?) and $MA = MO$.

5. $\triangle BAM$ is isos. (Why?) and $MA = AB$.

6. Isos. $\triangle BAM \sim$ isos. $\triangle BOA$. (Why?)

7. $\therefore \dfrac{OB}{AB} = \dfrac{AB}{BM}$, or $\overline{AB}^2 = OB \cdot BM$.

8. $\therefore \overline{MO}^2 = OB \cdot BM$; *i.e.* M divides the radius into extreme and mean ratio.

Construction. (Left to the student.)

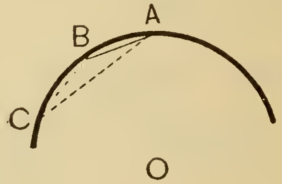
316. Corollary 1. — *By joining the alternate vertices of a regular inscribed decagon, a regular inscribed pentagon may be constructed.*

317. Corollary 2. — *A regular polygon of* 5, 10, 20, 40, $\cdots 5 \cdot 2^n$ *may be inscribed in a circle.*

How? Inscribe a regular polygon of 20 sides in a circle.

318. Corollary 3. — *By drawing tangents at the vertices of regular inscribed polygons with* 5, 10, 20, $\cdots 5 \cdot 2^n$ *sides, regular circumscribed polygons may be constructed with the same numbers of sides.*

319. Construction. — *Inscribe a regular pentedecagon in a given circle.*



Analysis. 1. Suppose the problem solved and AB one of the sides.

2. Then arc $AB = \frac{1}{15}$ of the circle.

3. Now, $\frac{1}{15} = \frac{1}{6} - \frac{1}{10}$.

4. $\therefore$ arc $AB =$ arc $AC -$ arc BC, where AC is the side of a regular hexagon and BC the side of a regular decagon.

Construction. (Left to the student.)

320. Corollary 1. — *A regular polygon of* 15, 30, 60, $\cdots 15 \cdot 2^n$ *sides may be inscribed in a circle.*

How? Inscribe a regular polygon of 30 sides.

321. Corollary 2. — *By drawing tangents at the vertices of regular inscribed pologons with* 15, 30, 60, ⋯ $15 \cdot 2^n$ *sides, regular circumscribed polygons with the same numbers of sides may be constructed.*

322. Summary. — The method of construction of regular polygons of 2^n, $3 \cdot 2^n$, $5 \cdot 2^n$, and $15 \cdot 2^n$ sides by use of circles has been shown. Until the beginning of the last century it was thought that these were the only regular polygons possible of construction by means of a *straightedge and compasses.* But Gauss proved that all those the number of whose sides may be expressed by $2^{2p} + 1$ may be constructed. This fact was published in 1801.

323. Metrical Relations. — 1. In any inscribed regular polygon

$$a = \sqrt{R^2 - \frac{s^2}{4}},$$

where $a =$ the apothem, $s =$ the side of the polygon, and $R =$ the radius of the circle. (Why?)

2. In a square, $a = \frac{s}{2} = \frac{R\sqrt{2}}{2}$. (Why?)

3. In a regular hexagon, $a = \frac{R\sqrt{3}}{2}$. (Why?)

4. In a regular decagon,

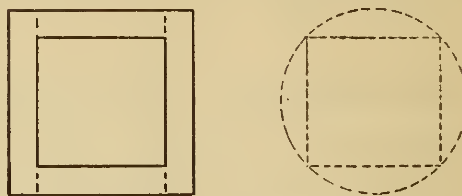
$$s = \frac{R(\sqrt{5} - 1)}{2} \text{ (Why?); } \quad a = \frac{R\sqrt{10 + 2\sqrt{5}}}{4} \text{ (Why?).}$$

EXERCISES

1. Construct a regular polygon of 5 sides; 6 sides; 8 sides; 10 sides.

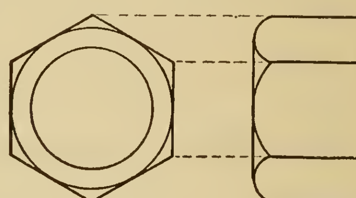
2. In making a waterwheel we have a square block of wood which is to be made into the form of a regular octagon by cutting off the four corners, and then buckets attached to each of the eight faces. Show how to cut off the corners accurately.

3. In making an Indian club, pieces of holly are glued to the faces of a square piece of gum wood. This is then turned up in a lathe, the holly producing oblong, oval light spots on the surface of the club. The thickness of the gum wood is 3 in. In order to get a club in which these oval spots just touch each other at the point of greatest thickness of the club, how thick must the pieces of holly be taken?

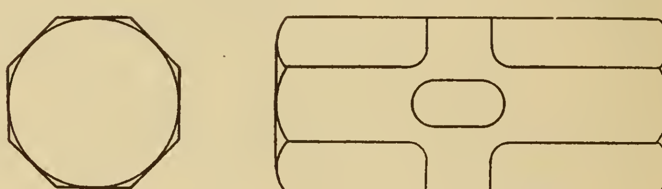


4. The stock from which tools and other pieces of metal generally are cut is cylindrical. A piece of iron is to be cut so that one end is a regular hexagon whose side is $\frac{5}{8}$ in. What size stock (diameter) must be selected to make it from?

5. Draw the face and edge views of the hexagonal nut of a bolt, the side of the nut to be one inch.

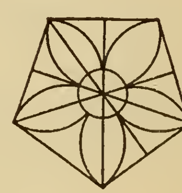


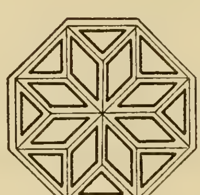
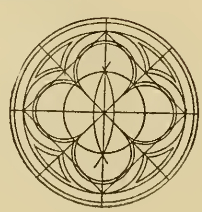
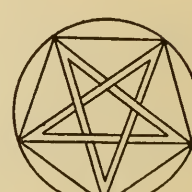
6. Draw the face and side views of an octagonal sledge hammer, the thickness of the hammer to be two inches from side to side.



7–13. Explain how the following ornamental designs are obtained, and make designs like them.





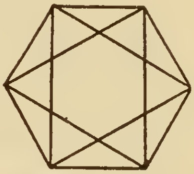




14. Construct regular pentagons, hexagons, octagons, and decagons, by drawing exterior angles. What is the size of the exterior angle in each?

NOTE. — Use protractor when needed.

15. The area of an inscribed regular hexagon is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

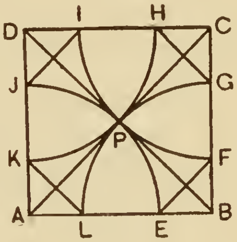
16. If the diagonals joining the alternate vertices of a regular hexagon are drawn, they form a second regular hexagon whose area is one third of that of the original hexagon.



17. If from any point within a regular polygon of n sides perpendiculars are drawn to the sides, the sum of these perpendiculars is equal to n times the apothem of the polygon.

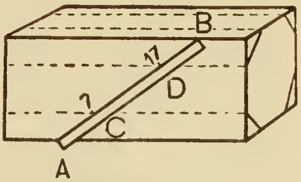
SUGGESTION. — Draw sects connecting the point with each vertex. Then get two expressions for the area of the polygon, and compare them.

18. The following method is much used in practical work of obtaining a regular octagon from a square. Draw the diagonals of the square, intersecting at P. With radius equal to AP, and centers A, B, C, D, draw arcs cutting the sides of the square at E, F, G, H, I, J, K, L. Draw EF, GH, IJ, and KL. Then $EFGHIJKL$ is a regular octagon. Prove it.



19. Carpenters, in order to cut a square piece of timber down to an octagonal shape, proceed as follows:

Place a 24-in. rule diagonally across the timber, the ends even with the edges at A and B, as in the figure. Then mark the points C and D at the 7-in. and 17-in. points on the rule. Through these points draw lines parallel to the edges of the timber. Repeat this on each face. The corners must be cut down to these lines to form an octagonal piece of timber. Is this method accurate or approximate? Prove your answer.



20. If R is the radius of a circle, the side of an inscribed equilateral triangle is $R\sqrt{3}$, its apothem is R, and its area is $\frac{3}{4}R^2\sqrt{3}$.

21. If R is the radius of a circle, the side of a circumscribed equilateral triangle is $2R\sqrt{3}$, and its area is $3R^2\sqrt{3}$.

22. If R is the radius of a circle, the side of a regular inscribed hexagon is R, and its area is $\frac{3}{2}R^2\sqrt{3}$.

23. If R is the radius of a circle, the side of a regular circumscribed hexagon is $\frac{2}{3}R\sqrt{3}$, and its area is $2R^2\sqrt{3}$.

THE MEASUREMENT OF THE CIRCLE

324. Area. — The **area of a circle** is the numerical measure (See §123) of the surface inclosed by it. As shown later, it can be computed only *approximately*.

The **area of any plane figure** is the numerical measure of the surface inclosed by its boundary line or lines.

NOTE. — The measurement of the circle requires the introduction here of certain theorems of a general nature about limits, whose applications are not difficult to understand, although rigorous proofs of them are too difficult for an elementary course in geometry. No proofs are here given. For a treatment of them see the "Stone-Millis Higher Algebra."

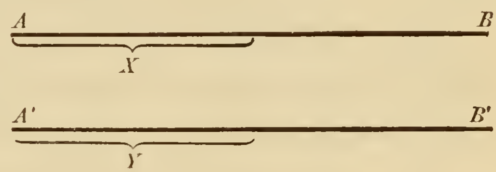
325. Variables and constants. — A **variable quantity,** or simply a **variable,** is a quantity whose value, under the condition or conditions imposed upon it, is constantly changing. Thus, the length of a chord revolving about one of its extremities constantly changes, and under that condition is a *variable.*

A **constant quantity,** or simply a **constant,** is a quantity whose value remains fixed.

326. Limits. — The **limit of a variable** is that *constant* which the variable continually approaches, and may be made to differ from by less than any fixed quantity however small, yet never reaches. As an example, suppose a point X on a sect AB to move from A toward B under the condi-

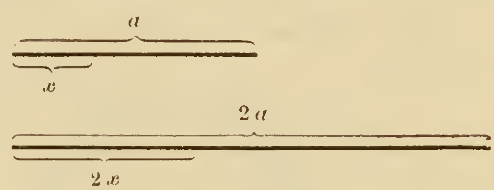
tion that during the first second it moves half the distance, and during the next second half the remaining distance, and so on. It is evident that by taking the number of seconds large enough X can be made to approach as near B as we please yet it can never be made to coincide with it. Hence the limiting position of X is B and the limit of the distance passed over, AX, is AB. This is expressed by $AX \doteq AB$. $AX \doteq AB$ is read "AX approaches AB as a limit."

327. Theorem. — *If two variables are always equal and each approaches a limit, the limits are equal.*



Thus, if two variables x and y are always equal and $x \doteq AB$ and $y \doteq A'B'$, then $AB = A'B'$.

328. Theorem. — *If the limit of a variable* **x** *is* **a**, *then the limit of* **kx** *is* **ka**, *where* **k** *is any constant.*



Thus, when $x \doteq a$, that is, when x approaches indefinitely near to a in value, yet does not reach it, $2x$ approaches indefinitely near to $2a$ in value, yet does not reach it, that is $2x \doteq 2a$.

329. Theorem. — *If the limit of a variable* **x** *is* **a**, *then the limit of* $\frac{x}{k}$ *is* $\frac{a}{k}$, *where* **k** *is any constant.*

This follows from § 328, because $\frac{x}{k}$ may be written $\frac{1}{k} \cdot x$, or a constant times x.

330. Assumptions. — Inscribe any regular polygon, as a square, in a circle. Bisect the arcs subtended by the sides, and inscribe a regular polygon with double the number of

sides. Continue this until an inscribed polygon is obtained with at least 16 or 32 sides. Likewise circumscribe a regular polygon about the circle and continue doubling the number of sides as above.

From the figure thus obtained it is assumed that

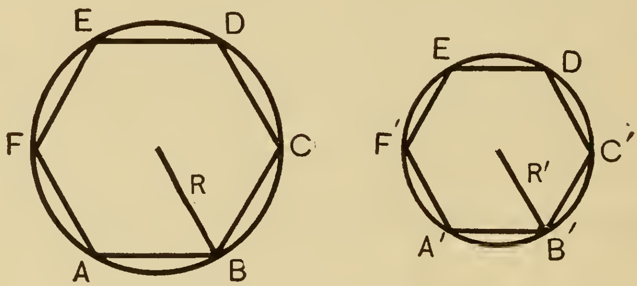
I. *The perimeters of the inscribed regular polygons are always less than the circle, and the perimeters of the circumscribed regular polygons are always greater than the circle.*

II. *The circle is the limit which the perimeters of both the inscribed and circumscribed regular polygons approach as the number of their sides indefinitely increases.*

III. *The area of the circle is the limit which the areas of both the inscribed and circumscribed regular polygons approach as the number of their sides indefinitely increases.*

IV. *The radius of the circle is the limit which the apothem of a regular inscribed polygon approaches as the number of sides indefinitely increases.*

331. Theorem. — *The ratio of a circle to its diameter is constant.*



Proof. 1. Let any two circles with radii R and R', and diameter D and D', respectively, have similar regular polygons inscribed in them, with perimeters P and P', respectively.

2. Then $\frac{P}{P'} = \frac{R}{R'} = \frac{2R}{2R'} = \frac{D}{D'}$; whence, $\frac{P}{D} = \frac{P'}{D'}$.

3. Now if the number of sides of the polygons is indefinitely increased, $P \doteq C$ and $P' \doteq C'$.

4. $$\therefore \frac{P}{D} \doteq \frac{C}{D} \text{ and } \frac{P'}{D'} \doteq \frac{C'}{D'}. \qquad (\S\ 329)$$

5. $$\therefore \frac{C}{D} = \frac{C'}{D'}. \qquad (\S\ 327)$$

That is, the ratio between the circle and its diameter is the same for any two circles, or is constant.

NOTE. — The constant ratio $\frac{C}{D}$ is designated by the Greek letter π (pī). Its value will be found in § 343.

332. Corollary 1. — $C = \pi D = 2 \pi R$, for $\frac{C}{D} = \pi$.

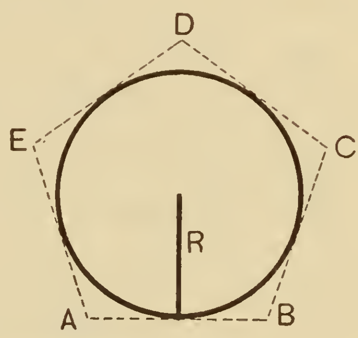
333. Corollary 2. — *Two circles have the same ratio as their diameters, or as their radii.*

For $$\frac{C}{C'} = \frac{\pi D}{\pi D'} = \frac{D}{D'} = \frac{R}{R'}.$$

334. Theorem. — *The area of a circle is equal to half the product of the radius and the circle.*

Hypothesis. R = radius, C = the circle, and S = the area of the circle.

Conclusion. $S = \frac{1}{2} R \cdot C$.



Proof. 1. Suppose the regular polygon $ABC \cdots$ circumscribed about the circle. Let S' = the variable area of the polygon as the number of sides of the polygon is increased indefinitely, and P = the variable perimeter.

2. Then for any polygon, $S' = \frac{1}{2} R \cdot P$. (§ 307)

3. But $S' \doteq S$ and $P \doteq C$. (§ 330)

4. $\therefore \frac{1}{2} R \cdot P \doteq \frac{1}{2} R \cdot C$. (§ 328)

5. Hence $S = \frac{1}{2} R \cdot C$. (§ 327)

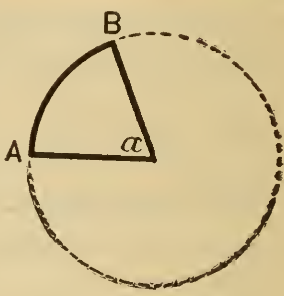
335. Corollary 1. — *The area of a circle equals* πR^2.

For, $S = \frac{1}{2} R \cdot C = \frac{1}{2} R \cdot 2\pi R = \pi R^2$.

336. Corollary 2. — *The areas of two circles are to each other as the squares of their radii, or as the squares of their diameters.*

For, $\dfrac{S}{S'} = \dfrac{\pi R^2}{\pi R'^2} = \dfrac{R^2}{R'^2} = \dfrac{D^2}{D'^2}$.

337. Corollary 3. — *If an arc AB of a circle subtends a central angle of* α *degrees, the length of the arc equals* $\dfrac{\alpha}{360} \cdot \pi D$.



For $\dfrac{\text{arc } AB}{C} = \dfrac{\alpha}{360}$; whence

$$\text{arc } AB = \frac{\alpha}{360} \cdot C = \frac{\alpha}{360} \cdot \pi D.$$

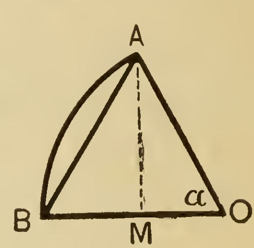
338. Definitions. — The figure formed by the arc of a circle and two radii drawn to its extremities is a **sector** of a circle.

The figure formed by an arc of a circle and the chord which subtends it is a **segment** of the circle.

339. Corollary 4. — *The area of a sector is equal to half the product of its radius by its arc.*

(Proof similar to that of the theorem.)

340.* Theorem. — *The area of a segment of a circle is equal to the area of the sector upon the same arc, less one half the product of the square of the radius by the sine of the central angle subtended by the arc.*



Hypothesis. The arc of segment AB subtends angle α at the center O, and the area of the segment is S. Also S' equals the area of sector ABO.

Conclusion. $S = S' - \frac{1}{2} R^2 \cdot \sin \alpha$.

Proof. 1. $S = S' - \triangle ABO$.

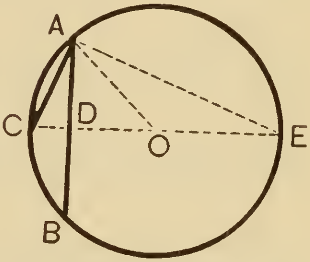
2. But $\triangle ABO = \frac{1}{2} BO \cdot AM = \frac{1}{2} R \cdot AM$.

3. $AM = OA \sin \alpha = R \sin \alpha$.

4. Hence $\triangle ABO = \frac{1}{2} R^2 \sin \alpha$.

5. $\therefore S = S' - \frac{1}{2} R^2 \sin \alpha$.

341. Computation. — *Given the side AB and the radius R of a regular inscribed polygon, find the side AC of the regular inscribed polygon of twice the number of sides.*



1. Draw diameter CE, and radius AO. Also draw AE.

2. Now $OD \perp AB$ at its middle point. (Why?)

3. $\therefore \overline{OD}^2 = R^2 - \frac{1}{4} \overline{AB}^2$.

4. $\therefore OD = \sqrt{R^2 - \frac{1}{4} \overline{AB}^2}$, and $CD = R - \sqrt{R^2 - \frac{1}{4} \overline{AB}^2}$.

5. Also $\overline{AC}^2 = CE \cdot CD = 2 R \cdot CD$.

6. $\therefore AC = \sqrt{2 R (R - \sqrt{R^2 - \frac{1}{4} \overline{AB}^2})}$

$$= \sqrt{R (2 R - \sqrt{4 R^2 - \overline{AB}^2})}.$$

342. Corollary. — *If $R = 1$, $S_n =$ the side of a regular inscribed polygon of n sides, and $S_{2n} =$ the side of a regular inscribed polygon of $2 n$ sides,*

$$S_{2n} = \sqrt{2 - \sqrt{4 - S_n^2}}.$$

343. Computation. — *Compute the value of π.*

1. Let $P_n =$ the perimeter of a regular polygon of n sides.

2. Let a regular hexagon be inscribed in a circle in which $R = 1$.

3. Then $S_6 = 1$. Hence by the formula in § 342,

$$S_{12} = \sqrt{2 - \sqrt{4 - 1^2}} = .51763809. \quad \therefore P_{12} = 6.21165708.$$

$$S_{24} = \sqrt{2 - \sqrt{4 - (.51763809)^2}} = .26105238. \quad \therefore P_{24} = 6.26525722.$$

$$S_{48} = \sqrt{2 - \sqrt{4 - (.26105238)^2}} = .13080626. \quad \therefore P_{48} = 6.27870041.$$

$$S_{96} = \sqrt{2 - \sqrt{4 - (.13080626)^2}} = .06543817. \quad \therefore P_{96} = 6.28206396.$$

$$S_{192} = \sqrt{2 - \sqrt{4 - (.06543817)^2}} = .03272346. \quad \therefore P_{192} = 6.28290510.$$

$$S_{384} = \sqrt{2 - \sqrt{4 - (.03272346)^2}} = .01636228. \quad \therefore P_{384} = 6.28311544.$$

$$S_{768} = \sqrt{2 - \sqrt{4 - (.01636228)^2}} = .00818121. \quad \therefore P_{768} = 6.28316941.$$

Thus, by continuing this computation, greater and greater perimeters are found, and hence closer and closer approximations to the circle. Using 6.28316941, the perimeter of a polygon of 768 sides, as approximately the circle, and dividing 6.28316941 by 2, the diameter, we get 3.14159, nearly, the approximate value of π.

Remark. — π is incommensurable. Its approximate value has been computed to over seven hundred decimal places. In practical work it is generally taken as 3.1416.

The ratio π has a long and interesting history. It was thought by some of the ancients to be 3, as shown in the Bible. Ahmes, an Egyptian, about 1700 B.C., gave a value of π that is equivalent to 3.1604 in modern notation. Archimedes, a Greek mathematician, born 287 B.C., proved that the value of π was between $3\frac{1}{7}$ and $3\frac{10}{71}$. Aryabhatta, a Hindu, born 476 A.D., found $\pi = 3.1416$. Many other values of π have been used by different people in the past.

EXERCISES

1. The radius of a circle is 3 in. Find the length of the circle; the area.

2. The area of a circle is 98 sq. in. Find the diameter.

3. In a circle whose radius is 8 ft., find the length of an arc whose central angle is 36°.

4. Find the area of a sector whose central angle is 40° and radius 12 in.

5. Find the area of a circular segment whose central angle is 56° and radius 9 in.

6. The diameters of two circles are 4 ft. and 9 ft., respectively. Find the ratio of their areas.

7. The central angle whose arc is equal to the radius is often used as the unit of measure of angles. It is called a *radian*. Find the number of degrees in a radian.

8. In the papyrus of Ahmes, an Egyptian, the area of a circle was found by subtracting from the diameter one ninth of its length and squaring the remainder. This was equivalent to using what value of π?

9. Archimedes, a Greek, who lived from about 287 B.C. to 212 B.C., and who was the greatest mathematician of antiquity, showed that the value of π was between $3\frac{1}{7}$ and $3\frac{10}{71}$. These numbers expressed decimally determine the true value of π to how many decimal places?

10. Show that the area of a circle is equal to that of a triangle whose base equals the length of the circle and altitude equals the radius. This was proved by Archimedes.

11. How many revolutions per mile does a 28-in. bicycle wheel make?

12. The boiler of an engine has 300 tubes, each 3 in. in diameter, for conducting the heat through the water. Find their total cross-sectional area.

13. A water tank for supplying locomotives on a railroad is cylindrical in form and 20 ft. in diameter. How long must a piece of strap iron be cut to make a band around it, allowing 20 in. for overlapping?

14. The earth is nearly 8000 mi. in diameter. What is the approximate length of the equator?

15. A canning company has cans made by the thousands for canning tomatoes. Each can is $4\frac{1}{4}$ in. in diameter and 5 in. high. Find the dimensions of the pieces of tin that must be cut to form the cylindrical surface of each can, allowing $\frac{1}{4}$ in. for a vertical seam.

16. How much belting does it require to run over two pulleys, each 24 in. in diameter and with their centers 16 ft. apart?

17. A grindstone of Ohio stone will stand a surface speed of 2500 ft. per minute. How many revolutions per minute will a $3\frac{1}{2}$-ft. stone safely stand?

18. An emery stone will safely stand a surface speed of 5500 ft. per minute. An emery grinder is to make 1500 revolutions per minute. What is the diameter of the largest wheel that may safely be used?

19. The driving pulley of an engine is 6 ft. 4 in., and makes 120 revolutions per minute. It is belted to a 24-in. pulley on the main shaft that runs the machinery of a mill. Find the speed of the shaft.

20. The steam pressure of an engine is indicated as 96 lb. per square inch. The cylinder of the engine is 20 in. in diameter. What is the total pressure on the piston?

21. A running track consists of two parallel straight stretches, each a quarter of a mile long, and two semicircular ends, each a quarter of a mile long at the inner curb. If two athletes run, one 5 ft. from the inner curb and the other 10 ft. from it, by how much is the second man handicapped?

22. The carrying capacity of five pipes, each 4 in. in diameter, is the the same, neglecting friction, as that of one pipe of what diameter?

23. How many 3-in. flues of an engine will equal in cross-sectional area a smokestack 3 ft. in diameter?

24. The boiler of an engine has 96 flues, each 4 in. in diameter. If the rule requires that the cross-sectional area of the smokestack shall be equal to the combined cross-sectional area of the flues, what must be the diameter of the smokestack for this engine?

25. In a mine there is a 1-in. pipe leading from each of four sumps. All of the water is to be collected into one sump, and just one pipe used. How large must this pipe be?

26. If it is customary, in iron turning, to allow a cutting speed at the rim of 40 ft. per minute, at what speed (revolutions per minute) should the lathe be driven for a piece of iron 2 in. in diameter?

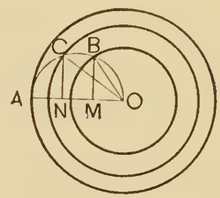
27. When the gauge shows a steam pressure of 100 lb. per square inch, what is the total pressure tending to blow the head out of an air drum 18 in. in inside diameter?

28. What is the propelling pressure exerted against the piston of a locomotive when the steam pressure is 98 lb. per square inch, the diameter of the cylinder being 28 in.?

29. Show how to go into the field and lay out a running track of the dimensions given in Exercise 21.

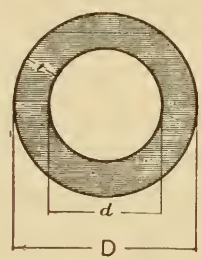
30. Through what angle must a 20-ft. rail of a railroad track be bent to fit a curve of a 400-ft. radius?

31. Divide a given circle into any number of equal parts (say 3) by drawing concentric circles.



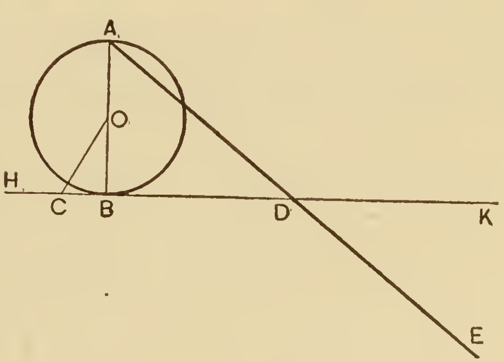
SUGGESTION. — Trisect the radius at M and N. Draw semicircle on AO as diameter. Erect perpendiculars MB and NC meeting semicircle at B and C. With centers at O and radii OB and OC, draw circles.

32. The following "cow and barn" problem is an old one: A cow is tethered at the end of a 100-ft. rope which is attached to a stake at the corner of a barn. The barn is 50 ft. square. Over how much area may the cow graze?



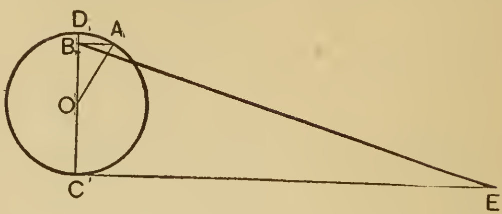
33. A convenient rule used in practical work for finding the area of a "hollow circle," or ring, is area $= \pi \times \dfrac{D+d}{2} \times t$. Establish this rule.

34. The following is Ceradini's approximate method of constructing a sect equal in length to a circle: Draw diameter AB and tangent HK at B. Draw OC making $\angle COB = 30°$. Make $CD = 3\ OB$. Draw AD, and prolong it, making $AE = 2\ AD$. Then AE is the required sect. Determine the accuracy of this construction by computing the ratio of AE to AB.



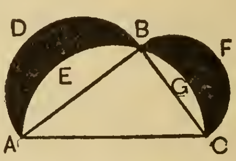
SUGGESTION. — Let $R =$ radius. Compute AB and AE in terms of R, then divide.

35. The approximate value of the circle may be obtained also as follows: Draw diameter CD. Draw central angle $AOB = 30°$. Draw $AB \perp CD$. Draw CE tangent at C and equal to $3\,CD$. Draw BE. Then BE equals the circle. Determine the accuracy of this construction.



36. In practical work, such as drafting constructions, the length of a circular arc is obtained by the formula $L = \dfrac{8c - C}{3}$, where L = length of arc, C = length of chord of arc, and c = length of chord of half the arc. This formula is very accurate for arcs of 60° or less. For larger arcs the arc is bisected and its half measured, then the result doubled. Show that when the arc subtends a central angle of 30°, the error is about 0.00006 of the radius. Draw an arc, and by use of the protractor measure the angle. Measure the chords and compute the arc by this formula. Check by computing the arc by the method in Ex. 3.

37. Semicircles are constructed upon the three sides of a right triangle, as in the figure. Show that the sum of the crescents $ADEB$ and $BFCG$ is equal to the triangle.



38. A crescent is bounded by a semicircle whose radius is A, and by the arc of another circle whose center is on the first arc produced. Find the area and perimeter of the crescent thus formed.

SYLLABUS OF PLANE GEOMETRY

(FOR REFERENCE OR REVIEW)

NOTE. The sections refer to the sections of the preceding pages of the text.

AXIOMS

§ 19. I. Things which are equal to the same thing, or to equal things, are equal to each other.

II. If equals are added to equals, the sums are equal.

III. If equals are subtracted from equals, the remainders are equal.

IV. If equals are multiplied by equals, the products are equal.

V. If equals are divided by equals, the quotients are equal.

VI. If equals are added to or subtracted from unequals, the results are unequal in the same order.

VII. The whole of a thing is equal to the sum of all its parts, and is greater than any one of its parts alone.

ASSUMPTIONS

§ 8. I. Any number of straight lines can be drawn through a point.

II. One and only one straight line can be drawn through two points. Or, two points determine a straight line.

III. The length of that portion of a straight line between two points is the shortest distance between the two points.

IV. A straight line can be produced indefinitely through either extremity. Or, a straight line is unlimited in extent.

V. Two straight lines can never intersect in more than one point. Or, two straight lines determine a point.

§ 17. If two adjacent angles are together equal to a straight angle, their exterior sides form a straight line.

§ 18. The half of a straight angle is a right angle.

§ 20. I. All straight angles are equal.

II. All right angles are equal.

§ 23. I. Complements of the same or of equal angles are equal.

II. Supplements of the same or of equal angles are equal.

§ 28. But one perpendicular can be drawn to a line through a given point.

§ 29. The perpendicular is the shortest sect that can be drawn from a point to a line.

§ 30. Two straight lines each perpendicular to the same straight line are parallel.

§ 31. If a sect connects two parallel lines, the alternate angles thus formed are equal.

§ 44. Through a given point only one line can be drawn parallel to a given line.

§ 98. The diagonals of any parallelogram intersect in a point which lies within the parallelogram.

§ 164. A circle of given radius may be described about a given point as center.

§ 169. A straight line cannot intersect a circle in more than two points.

§ 178. In the same circle, or in equal circles, equal central angles intercept equal arcs, and conversely.

§ 327. If two variables are always equal and each approaches a limit, the limits are equal.

§ 328. If the limit of a variable x is a, then the limit of kx is ka, where k is any constant.

§ 329. If the limit of a variable x is a, then the limit of $\frac{x}{k}$ is $\frac{a}{k}$, where k is any constant.

§ 330. I. The perimeters of the inscribed regular polygons are always less than the circle, and the perimeters of the circumscribed regular polygons are always greater than the circle.

II. The circle is the limit which the perimeter of both the inscribed and circumscribed regular polygons approach as the number of their sides indefinitely increases.

III. The area of the circle is the limit which the areas of both the inscribed and circumscribed regular polygons approach as the number of their sides indefinitely increases.

IV. The radius of the circle is the limit which the apothem of a regular inscribed polygon approaches as the number of sides indefinitely increases.

THEOREMS AND COROLLARIES

Angles, Perpendiculars, and Parallels

§ 25. If two straight lines intersect, the vertical angles thus formed are equal.

§ 32. If two parallel lines are cut by a transversal, the alternate exterior angles are equal.

§ 35. If two parallel lines are cut by a transversal, the corresponding angles are equal.

§ 36. If two parallel lines are cut by a transversal, the consecutive interior angles are supplementary.

§ 37. If two parallel lines are cut by a transversal, the consecutive exterior angles are supplementary.

§ 39. If two straight lines are cut by a transversal making the alternate interior angles equal, the lines are parallel.

§ 40. If two straight lines are cut by a transversal making the alternate exterior angles equal, the lines are parallel.

§ 41. If two straight lines are cut by a transversal making the corresponding angles equal, the lines are parallel.

§ 42. If two straight lines are cut by a transversal making the consecutive interior angles supplementary, the lines are parallel.

§ 43. If two straight lines are cut by a transversal making the consecutive exterior angles supplementary, the lines are parallel.

§ 46. Two straight lines parallel to the same straight line are parallel to each other.

§ 47. Angles having their sides parallel, each to each, and extending in the same direction, or in opposite directions from the vertex, are equal. If one pair of parallel sides extend in the same direction and the other pair in opposite directions from the vertex, the angles are supplementary.

§ 48. Angles having their sides perpendicular, each to each, and both acute or both obtuse, are equal. If one is acute and the other obtuse, they are supplementary.

Triangles

§ 59. The sum of all the angles of any triangle is equal to a straight angle.

§ 61. Cor. 1. If the values of two angles of a triangle are known, the value of the third angle is known.

§ 62. Cor. 2. If the value of an acute angle of a right triangle is known, the value of the other acute angle is also known.

§ 63. Cor. 3. In any triangle there must be at least two acute angles.

§ 64. Cor. 4. An exterior angle of any triangle is equal to the sum of the two opposite interior angles, and hence greater than either one of them.

§ 68. In any isosceles triangle, the angles opposite the equal sides are equal.

§ 69. Cor. 1. An equilateral triangle is also equiangular.

§ 70. Cor. 2. Each angle of an equilateral triangle is 60°.

§ 71. Cor. 3. In any isosceles triangle the bisector of the vertical angle is the perpendicular bisector of the base.

§ 72. Cor. 4. The median from the vertex of any isosceles triangle bisects the vertical angle.

§ 75. If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another, the triangles are congruent.

§ 76. Cor. 1. Two right triangles are congruent if the legs of one are equal respectively to the legs of the other.

§ 77. Cor. 2. In congruent triangles corresponding altitudes are equal.

§ 78. If two angles and a side of one triangle are equal respectively to two angles and the corresponding side of another, the triangles are congruent.

§ 79. Cor. 1. Two right triangles are congruent if the hypotenuse and an acute angle of one are equal to the hypotenuse and an acute angle of the other.

§ 80. Cor. 2. Two right triangles are congruent if a leg and an acute angle of one are equal to the corresponding leg and acute angle of the other.

§ 81. If three sides of one triangle are equal respectively to three sides of another, the two triangles are congruent.

§ 83. If two angles of a triangle are equal, the triangle is isosceles.

§ 84. Cor. An equiangular triangle is also equilateral.

§ 86. Cor. Two right triangles are congruent, if a leg and the hypotenuse of one equal a leg and the hypotenuse of the other.

§ 87. Two lines perpendicular respectively to two intersecting lines must meet.

Quadrilaterals: Polygons

§ 94. If two opposite sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

§ 95. In any parallelogram, (1) the opposite sides are equal; and (2) the opposite angles are equal.

§ 96. In any quadrilateral, if (1) the opposite sides are equal, or (2) the opposite angles are equal, the figure is a parallelogram.

§ 99. In any parallelogram, the diagonals bisect each other.

§ 100. If the diagonals of any quadrilateral bisect each other, the figure is a parallelogram.

§ 101. In any rectangle, the diagonals are equal.

§ 102. If the diagonals of a parallelogram are equal, the figure is a rectangle.

§ 103. Cor. In any right triangle, the middle point of the hypotenuse is equidistant from the vertices.

§ 104. The diagonals of a rhombus bisect each other at right angles.

§ 105. A sect drawn between the middle points of two sides of a triangle is parallel to the third side and equal to one half of it.

§ 106. If a straight line bisects one side of a triangle and is parallel to a second side, it bisects the third side also.

§ 108. The sum of the interior angles of a convex polygon of n sides is $n - 2$ straight angles.

§ 109. The sum of the exterior angles of any convex polygon is 2 straight angles.

Loci

§ 111. Every point on the perpendicular bisector of a sect is equidistant from the ends of the sect.

§ 112. Every point without the perpendicular bisector of a sect is unequally distant from the ends of the sect.

§ 114. The locus of all points at a given distance from a straight line is a pair of straight lines on opposite sides of the given line, parallel to it, and at the given distance from it.

§ 115. The locus of points within an angle and equidistant from the sides is the bisector of the angle.

SIMILAR TRIANGLES AND POLYGONS

§ 117. Any straight line parallel to a side of a triangle forms with the other two sides a triangle similar to the first.

§ 120. If a system of parallel lines divides any transversal into equal sects, it divides every transversal into equal sects.

§ 121. Cor. If a system of lines, parallel to a side of a triangle, divides one side of the triangle into equal sects, it divides the other side into equal sects also.

§ 124. A line parallel to one side of a triangle divides the other two sides into sects with the same ratio.

§ 125. Cor. A line parallel to a side of a triangle cuts off from the given triangle a triangle whose two sides including the common angle have the same ratio as the homologous sides of the given triangle.

§ 126. If a straight line divides two sides of a triangle into sects with equal ratios, it is parallel to the third side.

§ 127. Homologous sides of similar triangles have the same ratio.

§ 129. If two triangles have their corresponding sides proportional, the triangles are similar.

§ 130. If two triangles have an angle of one equal to an angle of the other and the including sides proportional, they are similar.

§ 136. In any two similar polygons, (1) the homologous angles are equal; (2) the homologous sides are proportional.

§ 137. Cor. In any two similar polygons the ratio of any two homologous lines equals the ratio of any other two homologous lines.

§ 138. Cor. In any two similar polygons, the perimeters have the same ratio as any two homologous sides.

Concurrent Lines of a Triangle

§ 149. In any triangle, the perpendiculars erected at the middle points of the sides are concurrent at a point equally distant from the three vertices.

§ 150. In any triangle, the three altitudes are concurrent.

§ 151. The medians of any triangle are concurrent in a point of trisection of each.

§ 152. The bisectors of the three angles of any triangle are concurrent in a point equally distant from the three sides.

Inequality

§ 153. If two of the sides of a triangle are unequal, the angles opposite are unequal, and the larger angle is opposite the longer side.

§ 154. If two of the angles of a triangle are unequal, the sides opposite are unequal, and the longer side is opposite the larger angle.

§ 155. In any triangle, any side is less than the sum of the other two sides and greater than their difference.

§ 156. In any triangle, the sum of two sects drawn from a point within the triangle to the extremities of any side is less than the sum of the other two sides.

§ 157. If two triangles have two sides of one equal to two sides of the other, but the included angle of the first greater than the included angle of the second, the sides opposite these angles are unequal, and the greater side lies opposite the greater angle.

§ 158. If two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, the angles opposite these sides are unequal, and the greater angle lies opposite the greater side.

Circles

§ 163. 1. A diameter is equal in length to the sum of two radii.

2. The distance of any point within the circle from the center is less than a radius.

3. The distance of any point without the circle from the center is greater than a radius.

4. All radii of the same circle are equal.

5. All diameters of the same circle are equal.

§ 165. A circle is determined by (1) the center and radius, (2) a diameter, or (3) three points not in the same straight line.

§ 166. Cor. 1. Two circles are congruent if the radius of one is equal to the radius of the other.

§ 167. Cor. 2. Conversely, if two circles are congruent, the radius of one is equal to the radius of the other.

§ 168. Cor. 3. The perpendicular bisector of a chord passes through the center of the circle.

§ 173. A straight line perpendicular to a radius at the point of intersection with a circle is a tangent.

§ 174. A tangent is perpendicular to the radius drawn to the point of tangency.

§ 176. Two tangents to a circle, from a point without the circle, are equal.

§ 177. Cor. The two tangents from a point without a circle to the circle make equal angles with the straight line joining the point to the center of the circle.

§ 180. Parallel secants of a circle intercept equal arcs.

§ 181. If a tangent and secant are parallel, they intercept equal arcs on the circle.

§ 182. If two tangents are parallel, they intercept equal arcs on the circle.

§ 183. If two arcs of the same circle or of equal circles are equal, they are subtended by equal chords, and conversely.

§ 184. Any diameter perpendicular to a chord bisects the chord and also the subtended arc.

§ 186. In the same circle, or in equal circles, (1) two equal chords are equidistant from the center, and (2) conversely, two chords equidistant from the center are equal.

§ 187. In the same circle, or in equal circles, two unequal chords are unequally distant from the center, and the greater chord is at the less distance.

§ 188. In the same circle, or in equal circles, two chords that are unequally distant from the center are unequal, and the chord at the less distance is the greater.

§ 189. Two circles cannot intersect in more than two points.

§ 191. The line of centers of two intersecting circles is perpendicular to their common chord and bisects it.

§ 193. If two circles are tangent to each other, the point of tangency lies on the line of centers.

§ 194. Two circles tangent to each other have a common internal tangent.

§ 195. If two circles are tangent to each other, the distance between their centers is equal to the sum or to the difference of their radii.

§ 196. In the same circle, or in equal circles, central angles have the same ratio as their intercepted arcs.

§ 197. Cor. A central angle has the same numerical measure as its intercepted arc.

§ 200. An inscribed angle has the same measure as one half of its intercepted arc.

§ 201. Cor. 1. All angles inscribed in the same segment are equal.

§ 202. Cor. 2. An angle inscribed in a semicircle is a right angle.

§ 206. An angle formed by a tangent and a chord from the point of tangency has the same measure as half the intercepted arc.

§ 208. The angle formed by two secants intersecting within the circle has the same measure as half the sum of the intercepted arcs.

§ 209. The angle formed by two secants, or a secant and tangent, or two tangents, intersecting without the circle has the same measure as half the difference between the intercepted arcs.

§ 210. Cor. The angle formed by two tangents has the same measure as the supplement of the smaller arc.

§ 212. The locus of the vertices of all angles which are equal to a given angle and whose sides pass through two given points is the arc of a circle terminating in the two points.

§ 214. A quadrilateral inscribed in a circle has its opposite angles supplementary.

§ 215. If the opposite angles of any quadrilateral are supplementary, the quadrilateral is inscriptible.

§ 216. In a circumscribed quadrilateral, the sum of two opposite sides equals the sum of the other two opposite sides.

Metrical Relations

§ 221. In any triangle, the bisector of an interior angle divides the opposite side internally into sects proportional to the adjacent sides, and the bisector of an exterior angle that intersects the opposite side divides it externally into sects proportional to the adjacent sides.

§ 222. If, in any triangle, a straight line from the vertex of an angle divides the opposite side internally into sects proportional to the adjacent sides, it is a bisector of that angle; and if a straight line from a vertex divides the opposite side externally into segments proportional to the adjacent sides, it is a bisector of the exterior angle at that vertex.

§ 224. If a pencil of lines cut two parallel lines, the corresponding sects intercepted on the parallels are proportional.

§ 225. If three straight lines intersect two parallels, making the corresponding intercepted sects proportional, the lines are either concurrent or parallel.

§ 227. In any right triangle, either leg is a mean proportional between the whole hypotenuse and its projection upon the hypotenuse.

§ 228. Cor. Any chord is a mean proportional between the diameter and its projection upon the diameter through one of its extremities.

§ 229. In any right triangle, the product of the legs equals the product of the hypotenuse and the altitude to the hypotenuse.

§ 230. In any right triangle, the altitude to the hypotenuse is the mean proportional between the two segments into which it divides the hypotenuse.

§ 231. Cor. The half of a chord perpendicular to a diameter is the mean proportional between the segments into which it divides the diameter.

§ 232. The squares of the legs of any right triangle are to each other as their projections upon the hypotenuse.

§ 233. In any right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.

§ 234. In any triangle, the square of a side opposite an acute angle is equal to the sum of the squares of the other two sides, less twice the product of one of these sides by the projection of the other upon it.

§ 235. In any obtuse triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides plus twice the product of one of these sides by the projection of the other upon it.

§ 236. An angle of a triangle is acute, right, or obtuse, according as the square of the opposite side is less than, equal to, or greater than the sum of the squares of the other two sides.

§ 237. In any triangle, if a median be drawn to the base, (1) the sum of the squares of the other two sides is equal to twice the square of the median plus twice the square of half the base; (2) the difference of the squares of the other two sides is equal to twice the product of the base and the projection of the median upon the base.

§ 238. In any triangle, the product of any two sides is equal to the product of the altitude to the third side by the diameter of the circumscribed circle.

§ 239. If through any point two secants be drawn to a circle, the product of the distances from the point to the two intersections on one secant is equal to the product of the distances from the point to the two intersections on the other.

§ 240. If from a point without a circle a secant and a tangent be drawn, the product of the distances from the point to the two intersections of the secant is equal to the square of the tangent.

§ 241. The square of the bisector of any angle of a triangle is equal to the product of the sides including the angle less the product of the sects of the third side made by the bisector.

Mensuration of Polygons

§ 255. Two rectangles having equal bases are to each other as their altitudes.

§ 256. Rectangles having equal altitudes are to each other as their bases.

§ 258. Any two rectangles are to each other as the products of their bases and their altitudes.

§ 259. The area of a rectangle is equal to the product of its base and altitude.

§ 260. Cor. The area of a square is equal to the square of one of its sides.

§ 261. Any parallelogram is equal to a rectangle having an equal base and an equal altitude.

§ 262. The area of a parallelogram is equal to the product of its base and altitude.

§ 263. Cor. 1. Parallelograms having equal bases and equal altitudes are equal.

§ 264. Cor. 2. Parallelograms are to each other as the products of their bases and altitudes.

§ 265. Cor. 3. Parallelograms having equal altitudes are to each other as their bases.

§ 266. Cor. 4. Parallelograms having equal bases are to each other as their altitudes.

§ 267. The area of any triangle is equal to one half the product of its base and altitude.

§ 268. Cor. 1. Triangles having equal bases and equal altitudes are equal.

§ 269. Cor. 2. Triangles are to each other as the products of their bases and altitudes.

§ 270. Cor. 3. Triangles having equal altitudes are to each other as their bases.

§ 271. Cor. 4. Triangles having equal bases are to each other as their altitudes.

§ 275. The area of any trapezoid is equal to one half the product of its altitude and the sum of its bases.

§ 276. Cor. The area of a trapezoid is equal to the product of its altitude and the sect joining the middle points of its non-parallel sides.

Comparison of Areas

§ 277. Two triangles having an angle of one equal to an angle of the other are to each other as the products of the sides including these angles.

§ 278. Two similar triangles are to each other as the squares of any two homologous sides.

§ 279. Two similar polygons are to each other as the squares of any two homologous sides.

§ 280. The square described upon the hypotenuse of a right triangle is equal to the sum of the squares described upon the two legs.

Regular Polygons: Circles

§ 291. An equilateral polygon inscribed in a circle is regular.

§ 292. A circle may be circumscribed about, or inscribed in, any regular polygon.

§ 293. Cor. 1. The inscribed and circumscribed circles of a regular polygon have the same center.

§ 295. Cor. 2. The angle at the center of any regular polygon is equal to four right angles divided by the number of sides of the polygon.

§ 296. Cor. 3. The radius drawn to any vertex of a regular polygon bisects the angle at the vertex.

§ 297. If any circle be divided into any number of equal arcs, (1) the chords of these arcs form a regular inscribed polygon; and (2) the tangents to the circle at the points of division form a regular circumscribed polygon.

§ 298. Cor. 1. If the vertices of a regular inscribed polygon are joined to the middle points of the arcs subtended by the sides of the polygon, the joining sects will form another regular inscribed polygon of twice the number of sides.

§ 299. Cor. 2. The tangents to a circle at the middle points of the arcs subtended by the sides of a regular inscribed polygon form a circumscribed regular polygon of double the number of sides.

§ 300. Cor. 3. The perimeter of an inscribed regular polygon is less than that of an inscribed regular polygon of double the number of sides.

§ 301. Cor. 4. The perimeter of a circumscribed regular polygon is greater than that of a circumscribed regular polygon of double the number of sides.

§ 302. The tangents to a circle at the middle points of the arcs subtended by the sides of a regular inscribed polygon form a regular circumscribed polygon whose sides are parallel to the corresponding sides of the inscribed polygon.

§ 303. Two regular polygons of the same number of sides are similar.

§ 304. Cor. The areas of two regular polygons of the same number of sides are to each other as the squares of any two homologous sides.

§ 305. The perimeters of two regular polygons of the same number of sides are to each other as their radii, and also as their apothems.

§ 306. Cor. The areas of two regular polygons of the same number of sides are to each other as the squares of their radii and of their apothems.

§ 307. The area of any regular polygon is equal to one half the product of its perimeter by its apothem.

§ 309. Cor. 1. A regular polygon of 4, 8, 16, ⋯ 2^n sides may be inscribed in a circle.

§ 310. Cor. 2. By drawing tangents at the vertices of regular inscribed polygons of 4, 8, 16, ⋯ 2^n sides, regular circumscribed polygons may be constructed with the same number of sides.

§ 312. Cor. 1. An equilateral triangle may be inscribed in a circle.

§ 313. Cor. 2. Regular polygons of 3, 6, 12, 24, ⋯ $3 \cdot 2^n$ sides may be inscribed in a circle.

§ 314. Cor. 3. By drawing tangents at the vertices of regular inscribed polygons with 3, 6, 12, ⋯ $3 \cdot 2^n$ sides, regular circumscribed polygons may be constructed with the same numbers of sides.

§ 316. Cor. 1. By joining the alternate vertices of a regular inscribed decagon, a regular inscribed pentagon may be constructed.

§ 317. Cor. 2. A regular polygon of 5, 10, 20, 40, ⋯ $5 \cdot 2^n$ sides may be inscribed in a circle.

§ 318. Cor. 3. By drawing tangents at the vertices of regular inscribed polygons with 5, 10, 20, ⋯ $5 \cdot 2$ sides, regular circumscribed polygons may be constructed with the same numbers of sides.

§ 320. A regular polygon of 15, 30, 60, ⋯ $15 \cdot 2^n$ sides may be inscribed in a circle.

§ 321. Cor. 2. By drawing tangents at the vertices of regular inscribed polygons with 15, 30, 60, ⋯ $15 \cdot 2^n$ sides, regular circumscribed polygons with the same numbers of sides may be constructed.

§ 331. The ratio of a circle to its diameter is constant.

§ 332. Cor. 1. $C = \pi D = 2\pi R$.

§ 333. Cor. 2. Two circles have the same ratio as their diameters, or as their radii.

§ 334. The area of a circle is equal to half the product of the radius and the circle.

§ 335. Cor. 1. The area of a circle equals πR^2.

§ 336. Cor. 2. The areas of two circles are to each other as the squares of their radii, or as the squares of their diameters.

§ 337. Cor. 3. If an arc AB of a circle subtends a central angle of α degrees, the length of the arc equals $\frac{\alpha}{360} \cdot \pi D$.

§ 339. Cor. 4. The area of a sector is equal to half the product of its radius by its arc.

§ 340.* The area of a segment of a circle is equal to the area of the sector upon the same arc, less one half the product of the square of the radius by the sine of the central angle subtended by the arc.

§ 342. Cor. If $R = 1$, S_n = the side of a regular inscribed polygon of n sides, and S_{2n} = the side of a regular inscribed polygon of $2n$ sides,

$$S_{2n} = \sqrt{2 - \sqrt{4 - S_n^2}}.$$

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